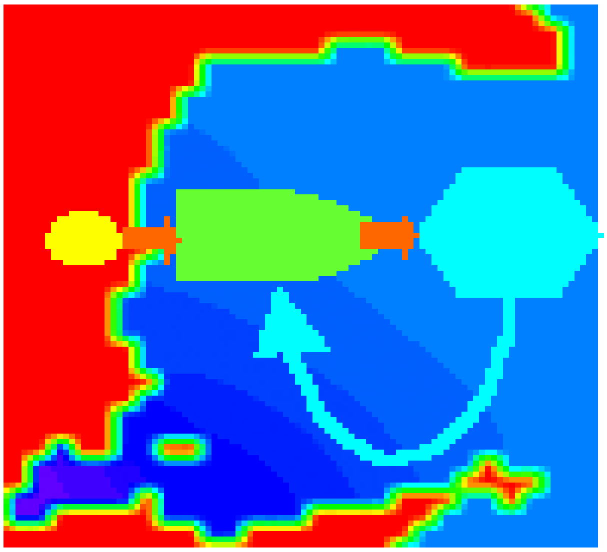
A Three dimensional hydrodynamic model implemented with EcoDynamo

Implemetação de um modelo hidrodinâmico tridimensional com o EcoDynamo.



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1 Introduction

This report presents a three dimensional hydrodynamic and transport model developed at University Fernando Pessoa by the author, over the period 2006-2008. This model is a follow-up of a two dimensional model developed previously within the scope of two European projects (Duarte et al., 2003, 2005, 2008). It was applied first in the work of two post-graduated students – one Msc and one PhD student, for the Alqueva reservoir and the Douro estuary, respectively. The model was implemented with EcoDynamo – an object oriented modelling software (Pereira & Duarte, 2005; Pereira et al., 2006). In this report, all the equations and respective numerical resolution are presented, together with some explanations.

2 General equations

The model is based on the horizontal equations of motion (1 and 2), the equation of continuity (3 and 4), the transport equation (5) and on the hydrostatic assumption (Pond & Pickard, 1983).

$$\frac{du_x}{dt}+\frac{\partial(uu)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}-\nu_x\frac{\partial^2 u}{\partial x^2}-\nu_y\frac{\partial^2 u}{\partial y^2}-\nu_z\frac{\partial^2 u}{\partial z^2}=-\frac{\partial P}{\rho\partial x}-\frac{Cf\,|u|u}{H}+\frac{\partial\tau_x}{\rho\partial z}+fv_y \quad \textbf{(1)}$$

$$\frac{dv_y}{dt}+\frac{\partial(vv)}{\partial y}+\frac{\partial(vu)}{\partial x}+\frac{\partial(vw)}{\partial z}-\nu_x\frac{\partial^2 v}{\partial x^2}-\nu_y\frac{\partial^2 v}{\partial y^2}-\nu_z\frac{\partial^2 v}{\partial z^2}=-\frac{\partial P}{\rho\partial y}-\frac{Cf\,|v|v}{H}+\frac{\partial\tau_y}{\rho\partial z}-fu_x \quad \textbf{(2)}$$

$$\frac{d\xi}{dt}=\frac{\partial u_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial w_z}{\partial z} \qquad \textbf{(3)}$$

$$\frac{\partial u_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial w_z}{\partial z}=0 \qquad \textbf{(4)}$$

$$\frac{dS}{dt}+\frac{\partial(uS)}{\partial x}+\frac{\partial(vS)}{\partial y}+\frac{\partial(wS)}{\partial z}=\frac{\partial\left(A_x\frac{\partial S}{\partial x}\right)}{\partial x}+\frac{\partial\left(A_y\frac{\partial S}{\partial y}\right)}{\partial y}+\frac{\partial\left(A_z\frac{\partial S}{\partial z}\right)}{\partial z} \qquad \textbf{(5)}$$

Where,

u, v and w - Current speed (East-West(x), North- South(y) and Bottom-Up(z), respectively) (m s^{-1}); P – Pressure (N m^{-2}); ρ – Density (kg m^{-3}); ν - Eddy diffusivity (East-West(x), North- South(y) and Bottom-Up(z), respectively) (m^2 s^{-1}); *Cf* – rugosity coefficient (dimensionless); τ - wind stress (N m^{-2}); *f* – Coriolis parameter; ξ - Surface elevation (m); H- Depth (m); S – concentration of any dissolved or particulate property (kg m^{-3}); A – diffusivity of the mentioned property along each of the three dimensions m^2 s^{-1}).

Equation 3 is applied at the surface, where flow divergence/convergence may change surface elevation. Equation 4 is applied below the surface, where the balance of inflows and outflows at any control volume must be zero.

3 Numerical solutions

The numerical solutions were obtained following a finite difference scheme, based on a Cartesian coordinate system (Fig. 1). Hereafter, superscripts t-1, t and t+1, refer the time step to which the value being used corresponds. i or j-1, -1/2, +1/2 or +1 represent the position in the model grid where the variable is evaluated. i, j and k refer grid cell coordinates along west-east, south-north and bottom-up directions, respectively.

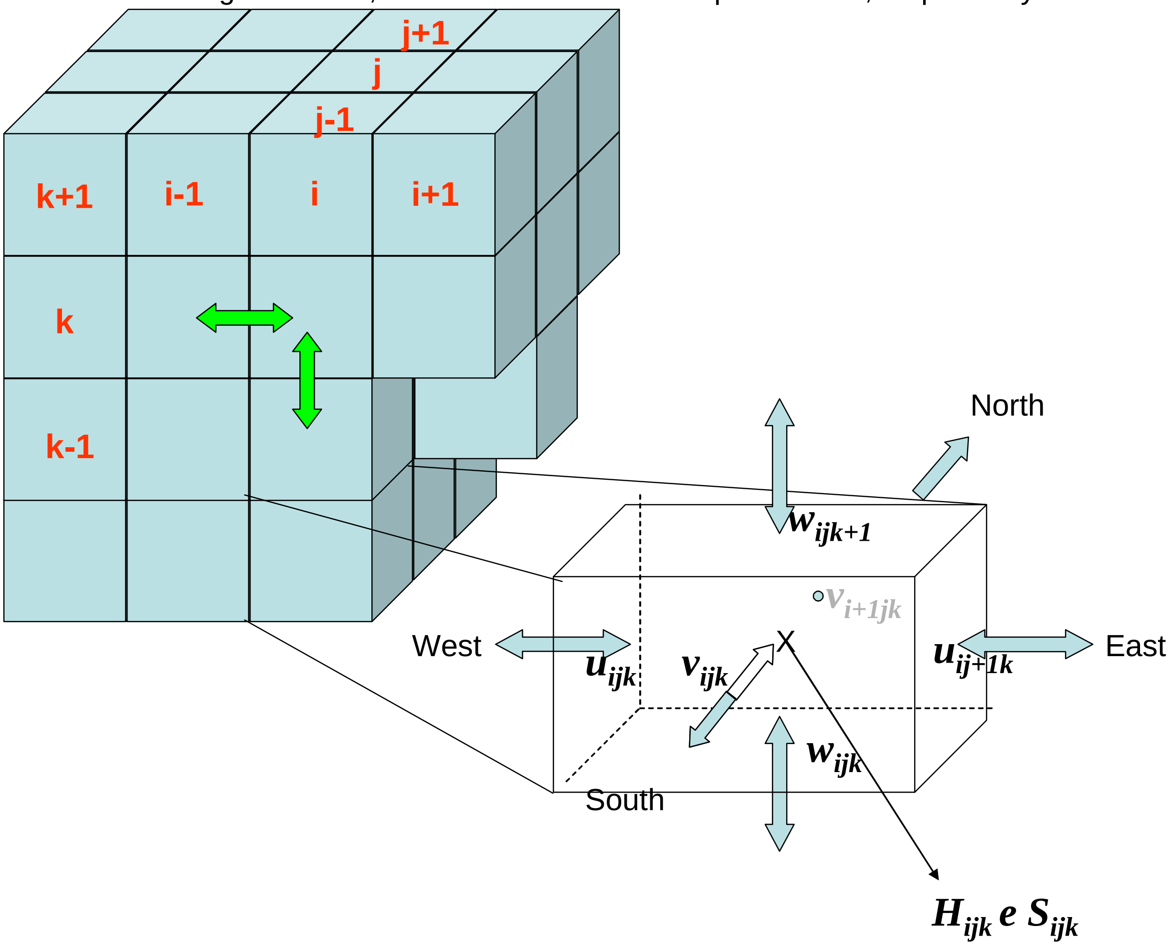

Figure 1 – A simplified view of the three dimensional model grid (upper left) showing the usage of indexes i, j and k. Scalar quantities are calculated at the centre of each cell, whereas vector quantities are calculated at the interfaces between cells, as shown in the amplified grid cell (lower right) .

In this model, a semi-implicit numerical scheme is used for stability. The generic algorithm is:

$$v_{ijk}^{t+1} \rightarrow \xi_{ij}^{t+1/2} \rightarrow u_{ijk}^{t+1} \rightarrow w_{ijk}^{t+1} \rightarrow V_{ijk}^{t+1} and\, \xi_{ij}^{t+1} \rightarrow S_{ij}^{t+1}$$

One horizontal velocity component is calculated explicitly and together with an implicit calculation of the other horizontal velocity component, embedded in the equation of continuity, used to estimate water levels. Using these water levels in the barotropic pressure term, the implicit horizontal velocity component is calculated implicitly. Afterwards, vertical velocities are calculated with the continuity equation from bottom till the surface. At the surface layer, the numerical solution of equation 3 is used to calculate a final estimate for the water level. Calculated velocities are then used in the transport equation to calculate the concentration of dissolved or particulate substances. In the next (**semi**) time step, the explicit velocity term of the previous one becomes implicit and the implicit becomes explicit and so forth.

3.1 Numerical solutions for the equations of motion

Equations 6 and 7 are the numerical version of equations 1 and 2.

$$\frac{\Delta u_x}{dt}+\frac{\Delta(uuH\Delta y)}{H\Delta y\partial x}+\frac{\Delta(uvH\Delta x)}{H\Delta x\partial y}+\frac{\Delta(uw\Delta y)}{\Delta y\partial z}-\nu_x\frac{\Delta^2 u}{\Delta x^2}-\nu_y\frac{\partial^2 v}{\partial y^2}-\nu_z\frac{\Delta^2 w}{\Delta z^2}=$$

$$-g\left(\frac{\Delta\xi}{\Delta x}+\frac{\int_{z=0}^{z}\rho_z\partial z}{\rho\Delta x}\right)-\frac{Cf|u|u}{H}+\frac{\Delta\tau_x}{\rho\Delta z}+f\,v_x \qquad \textbf{(6)}$$

$$\frac{\Delta v_x}{dt}+\frac{\Delta(vvH\Delta x)}{H\Delta x\partial y}+\frac{\Delta(vuH\Delta y)}{H\Delta y\partial x}+\frac{\Delta(vw\Delta y)}{\Delta y\partial z}-\nu_x\frac{\Delta^2 u}{\Delta x^2}-\nu_y\frac{\partial^2 v}{\partial y^2}-\nu_z\frac{\Delta^2 w}{\Delta z^2}=$$

$$-g\left(\frac{\Delta\xi}{\Delta y}+\frac{\int_{z=0}^{z}\rho_z\partial z}{\rho\Delta y}\right)-\frac{Cf|v|v}{H}+\frac{\Delta\tau_y}{\rho\Delta z}-f\,u_x \qquad \textbf{(7)}$$

3.1.1 West-East velocity component

In the following pages, the West-East velocity component (hereafter referred as u component) will be solved implicitly considering intermediate, bottom and surface layers to emphasize some differences related to the presence of different borders. The explicit version of each equation will be presented first, followed by the implicit solution.

Intermediate layers:

$$
\begin{aligned}
&\frac{u_{ijk}^{t+1}-u_{ijk}^{t}}{\Delta t}+\\
&\frac{1}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left[\begin{array}{l}
\dfrac{H_{ijk}^{t}\left(u_{ij+\frac{1}{2}k}^{t}+\left|u_{ij+\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}-H_{ij-1k}^{t}\left(u_{ij-\frac{1}{2}k}^{t}+\left|u_{ij-\frac{1}{2}k}^{t}\right|\right)u_{ij-1k}^{t}}{\Delta x_{ij-\frac{1}{2}k}}+\\
\dfrac{H_{ijk}^{t}\left(u_{ij+\frac{1}{2}k}^{t}-\left|u_{ij+\frac{1}{2}k}^{t}\right|\right)u_{ij+1k}^{t}-H_{ij-1k}^{t}\left(u_{ij-\frac{1}{2}k}^{t}-\left|u_{ij-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}}{\Delta x_{ij-\frac{1}{2}k}}
\end{array}\right]\\
&+\frac{1}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left[\begin{array}{l}
\dfrac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}+\left|v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}+\left|v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{i-1jk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}+\\
\dfrac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}-\left|v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{i+1jk}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}-\left|v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}
\end{array}\right]\\
&+\left[\begin{array}{l}
\dfrac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}+\left|w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}\right|\right)u_{ijk}^{t}-\left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}+\left|w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}\right|\right)u_{ijk-1}^{t}}{2H_{ij-\frac{1}{2}k}^{t}}+\\
\dfrac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}-\left|w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}\right|\right)u_{ijk+1}^{t}-\left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}-\left|w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}\right|\right)u_{ijk}^{t}}{2H_{ij-\frac{1}{2}k}^{t}}
\end{array}\right]=\\
&-g\left(\frac{\xi_{ij}^{t+\frac{1}{2}}-\xi_{ij-1}^{t+\frac{1}{2}}}{\Delta x_{ij-\frac{1}{2}k}}+\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{ij-1k}\right)\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)/2}{\left(\rho_{ijk}+\rho_{ij-1k}\right)/2\,\Delta x_{ij-\frac{1}{2}k}}\right)\\
&+\frac{\nu_{x}}{\Delta x_{ijk}^{2}}\left(u_{ij+1k}^{t}-u_{ijk}^{t}\right)-\frac{\nu_{x}}{\Delta x_{ij-1k}^{2}}\left(u_{ijk}^{t}-u_{ij-1k}^{t}\right)\\
&+\frac{\nu_{y}}{\Delta y_{i+\frac{1}{2}jk}^{2}}\left(u_{i+1jk}^{t}-u_{ijk}^{t}\right)-\frac{\nu_{y}}{\Delta y_{i-\frac{1}{2}jk}^{2}}\left(u_{ijk}^{t}-u_{i-1jk}^{t}\right)\\
&+\frac{\nu_{ijk+1}}{\Delta z_{ij-\frac{1}{2}k+\frac{1}{2}}^{2}}\left(u_{ijk+1}^{t+1}-u_{ijk}^{t+1}\right)-\frac{\nu_{ijk}}{\Delta z_{ij-\frac{1}{2}k-\frac{1}{2}}^{2}}\left(u_{ijk}^{t+1}-u_{ijk-1}^{t+1}\right)\\
&+f\,v_{ijk}^{*}
\end{aligned}
\tag{8}
$$

Equation (8) may be solved explicitly for u or implicitly as described below. In this case, it must be rearranged in order to obtain a tridiagonal system with coefficients *a*, *b*, *c* and *r*, that may be solved for *u* in *t+1* (9 and 10). The system is represented in (10). It may be solved by matrix inversion and multiplication using very efficient algorithms, like Tridag (Press et al., 1992).

$$
\begin{aligned}
&-\frac{\Delta t \nu_{ijk}}{\Delta z^2_{ij-\frac{1}{2}k-\frac{1}{2}}} u^{t+1}_{ijk-1} + u^{t+1}_{ijk} + \frac{\Delta t \nu_{ijk}}{\Delta z^2_{ij-\frac{1}{2}k+\frac{1}{2}}} u^{t+1}_{ijk} + \frac{\Delta t \nu_{ijk+1}}{\Delta z^2_{ij-\frac{1}{2}k+\frac{1}{2}}} u^{t+1}_{ijk} - \frac{\Delta t \nu_{ijk+1}}{\Delta z^2_{ij-\frac{1}{2}k+\frac{1}{2}}} u^{t+1}_{ijk+1} = u^t_{ijk} \\
&-\frac{\Delta t}{\left(H^t_{ijk}+H^t_{ij-1k}\right)} \left[\frac{H^t_{ijk}\left(u^t_{ij+\frac{1}{2}k}+\left|u^t_{ij+\frac{1}{2}k}\right|\right)u^t_{ijk} - H^t_{ij-1k}\left(u^t_{ij-\frac{1}{2}k}+\left|u^t_{ij-\frac{1}{2}k}\right|\right)u^t_{ij-1k}}{\Delta x_{ij-\frac{1}{2}k}} + \frac{H^t_{ijk}\left(u^t_{ij+\frac{1}{2}k}-\left|u^t_{ij+\frac{1}{2}k}\right|\right)u^t_{ij+1k} - H^t_{ij-1k}\left(u^t_{ij-\frac{1}{2}k}-\left|u^t_{ij-\frac{1}{2}k}\right|\right)u^t_{ijk}}{\Delta x_{ij-\frac{1}{2}k}} \right] \\
&-\frac{\Delta t}{\left(H^t_{ijk}+H^t_{ij-1k}\right)} \left[\frac{H^t_{i+\frac{1}{2}j-\frac{1}{2}k}\left(v^t_{i+\frac{1}{2}j-\frac{1}{2}k}+\left|v^t_{i+\frac{1}{2}j-\frac{1}{2}k}\right|\right)u^t_{ijk} - H^t_{i-\frac{1}{2}j-\frac{1}{2}k}\left(v^t_{i-\frac{1}{2}j-\frac{1}{2}k}+\left|v^t_{i-\frac{1}{2}j-\frac{1}{2}k}\right|\right)u^t_{i-1jk}}{\Delta y_{i-\frac{1}{2}jk}} + \frac{H^t_{i+\frac{1}{2}j-\frac{1}{2}k}\left(v^t_{i+\frac{1}{2}j-\frac{1}{2}k}-\left|v^t_{i+\frac{1}{2}j-\frac{1}{2}k}\right|\right)u^t_{i+1jk} - H^t_{i-\frac{1}{2}j-\frac{1}{2}k}\left(v^t_{i-\frac{1}{2}j-\frac{1}{2}k}-\left|v^t_{i-\frac{1}{2}j-\frac{1}{2}k}\right|\right)u^t_{ijk}}{\Delta y_{i-\frac{1}{2}jk}} \right] \\
&-\Delta t \left[\frac{\left(w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k+1}+\left|w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k+1}\right|\right)u^t_{ijk} - \left(w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k}+\left|w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k}\right|\right)u^t_{ijk-1}}{2H^t_{ij-\frac{1}{2}k}} + \frac{\left(w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k+1}-\left|w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k+1}\right|\right)u^t_{ijk+1} - \left(w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k}-\left|w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k}\right|\right)u^t_{ijk}}{2H^t_{ij-\frac{1}{2}k}} \right] \\
&-g\Delta t \left(\frac{\xi^{t+\frac{1}{2}}_{ij} - \xi^{t+\frac{1}{2}}_{ij-1}}{\Delta x_{ij-\frac{1}{2}}} + \frac{\sum_{k=0}^{k=surflayer} \left(\rho_{kj}-\rho_{kj-1}\right)\left(H^t_{kj}+H^t_{kj-1}\right)/2}{\left(\rho_{ij}+\rho_{ij-1}\right)/2\,\Delta x_{ij-\frac{1}{2}}} \right) \\
&+\frac{\nu_x}{\Delta x^2_{ijk}}\left(u^t_{ij+1k}-u^t_{ijk}\right)\Delta t - \frac{\nu_x}{\Delta x^2_{ij-1k}}\left(u^t_{ijk}-u^t_{ij-1k}\right)\Delta t \\
&+\frac{\nu_y}{\Delta y^2_{i+\frac{1}{2}jk}}\left(u^t_{i+1jk}-u^t_{ijk}\right)\Delta t - \frac{\nu_y}{\Delta y^2_{i-\frac{1}{2}jk}}\left(u^t_{ijk}-u^t_{i-1jk}\right)\Delta t + f v^*_{ijk}
\end{aligned}
$$

(9)

$$a_{jki}u_{ijk-1}^{t+1}+b_{jki}u_{ijk}^{t+1}+c_{jki}u_{ijk+1}^{t+1}=r_{ijk}$$

$$a_{jki}=-\frac{\Delta t\nu_{ijk}}{\Delta z^{2}_{ij-\frac{1}{2}k-\frac{1}{2}}}$$

$$b_{jki}=1+\frac{\Delta t\nu_{ijk}}{\Delta z^{2}_{ij-\frac{1}{2}k-\frac{1}{2}}}+\frac{\Delta t\nu_{ijk+1}}{\Delta z^{2}_{ij-\frac{1}{2}k+\frac{1}{2}}} \qquad \textbf{(10)} \qquad \begin{vmatrix} b_{jki} & c_{jki} & \cdots & \cdots \\ a_{jki} & b_{jki} & c_{jki} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & a_{jki+n} & b_{jki+n} \end{vmatrix} \begin{vmatrix} u_{ijk}^{t+1} \\ u_{ijk+1}^{t+1} \\ \cdots \\ \ldots \end{vmatrix} = \begin{vmatrix} r_{ijk} \\ \cdots \\ \cdots \\ \cdots \end{vmatrix}$$

$$c_{jki}=-\frac{\Delta t\nu_{ijk+1}}{\Delta z^{2}_{ij-\frac{1}{2}k+\frac{1}{2}}}$$

Bottom layer:

At the bottom layer it is necessary to consider bottom drag (last term on the right side of equation 11).

$$\begin{aligned}
&\frac{u_{ijk}^{t+1}-u_{ijk}^{t}}{\Delta t} \\
&+\frac{1}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left[\frac{H_{ijk}^{t}\left(u_{ij+\frac{1}{2}k}^{t}+\left|u_{ij+\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}-H_{ij-1k}^{t}\left(u_{ij-\frac{1}{2}k}^{t}+\left|u_{ij-\frac{1}{2}k}^{t}\right|\right)u_{ij-1k}^{t}}{\Delta x_{ij-\frac{1}{2}k}}+\frac{H_{ijk}^{t}\left(u_{ij+\frac{1}{2}k}^{t}-\left|u_{ij+\frac{1}{2}k}^{t}\right|\right)u_{ij+1k}^{t}-H_{ij-1k}^{t}\left(u_{ij-\frac{1}{2}k}^{t}-\left|u_{ij-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}}{\Delta x_{ij-\frac{1}{2}k}}\right] \\
&+\frac{1}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left[\frac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}+\left|v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}+\left|v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{i-1jk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}+\frac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}-\left|v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{i+1jk}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}-\left|v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}\right] \\
&+\left[\frac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}+\left|w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}\right|\right)u_{ijk}^{t}-\left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}+\left|w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}\right|\right)u_{ijk-1}^{t}}{2H_{ij-\frac{1}{2}k}^{t}}+\frac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}-\left|w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}\right|\right)u_{ijk+1}^{t}-\left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}-\left|w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}\right|\right)u_{ijk}^{t}}{2H_{ij-\frac{1}{2}k}^{t}}\right]= \\
&-g\left(\frac{\xi_{ij}^{t+\frac{1}{2}}-\xi_{ij-1}^{t+\frac{1}{2}}}{\Delta x_{ij-\frac{1}{2}k}}+\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{ij-1k}\right)\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)/2}{\left(\rho_{ijk}+\rho_{ij-1k}\right)/2\,\Delta x_{ij-\frac{1}{2}k}}\right) \\
&+\frac{\nu_{x}}{\Delta x_{ijk}^{2}}\left(u_{ij+1k}^{t}-u_{ijk}^{t}\right)-\frac{\nu_{x}}{\Delta x_{ij-1k}^{2}}\left(u_{ijk}^{t}-u_{ij-1k}^{t}\right) \\
&+\frac{\nu_{y}}{\Delta y_{i+\frac{1}{2}jk}^{2}}\left(u_{i+1jk}^{t}-u_{ijk}^{t}\right)-\frac{\nu_{y}}{\Delta y_{i-\frac{1}{2}jk}^{2}}\left(u_{ijk}^{t}-u_{i-1jk}^{t}\right) \\
&+\frac{\nu_{ijk+1}}{\Delta z_{ij-\frac{1}{2}k+\frac{1}{2}}^{2}}\left(u_{ijk+1}^{t+1}-u_{ijk}^{t+1}\right) \qquad \textbf{(11)} \\
&-\frac{1}{H_{ijk}^{t}+H_{ij-1k}^{t}}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left\|u_{ijk}^{t}\right\|u_{ijk}^{t+1}+f\,v_{ijk}^{*}
\end{aligned}$$

Again, this equation is rearranged in order to obtain a tridiagonal system (12 and 13). However, due to stability criteria (see above) the bottom drag term is solved implicitly, reason why it is on the left side of 12.

$$
\begin{aligned}
& u_{ijk}^{t+1}\left(1+\frac{\Delta t}{H_{ijk}^{t}+H_{ij-1k}^{t}}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u_{ijk}^{t}\right|\right)+u_{ijk}^{t+1}\frac{\Delta t\nu_{ijk+1}}{\Delta z_{ij-1/2k+1/2}^{2}}-u_{i+1j}^{t+1}\frac{\Delta t\nu_{ijk+1}}{\Delta z_{ij-1/2k+1/2}^{2}}= \\
& u_{ijk}^{t}-\frac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left[\frac{H_{ijk}^{t}\left(u_{ij+1/2k}^{t}+\left|u_{ij+1/2k}^{t}\right|\right)u_{ijk}^{t}-H_{ij-1k}^{t}\left(u_{ij-1/2k}^{t}+\left|u_{ij-1/2k}^{t}\right|\right)u_{ij-1k}^{t}}{\Delta x_{ij-1/2k}}+\frac{H_{ijk}^{t}\left(u_{ij+1/2k}^{t}-\left|u_{ij+1/2k}^{t}\right|\right)u_{ij+1k}^{t}-H_{ij-1k}^{t}\left(u_{ij-1/2k}^{t}-\left|u_{ij-1/2k}^{t}\right|\right)u_{ijk}^{t}}{\Delta x_{ij-1/2k}}\right] \\
& -\frac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left[\frac{H_{i+1/2j-1/2k}^{t}\left(v_{i+1/2j-1/2k}^{t}+\left|v_{i+1/2j-1/2k}^{t}\right|\right)u_{ijk}^{t}-H_{i-1/2j-1/2k}^{t}\left(v_{i-1/2j-1/2k}^{t}+\left|v_{i-1/2j-1/2k}^{t}\right|\right)u_{i-1jk}^{t}}{\Delta y_{i-1/2jk}}+\frac{H_{i+1/2j-1/2k}^{t}\left(v_{i+1/2j-1/2k}^{t}-\left|v_{i+1/2j-1/2k}^{t}\right|\right)u_{i+1jk}^{t}-H_{i-1/2j-1/2k}^{t}\left(v_{i-1/2j-1/2k}^{t}-\left|v_{i-1/2j-1/2k}^{t}\right|\right)u_{ijk}^{t}}{\Delta y_{i-1/2jk}}\right] \\
& -\Delta t\left[\frac{\left(w_{ij-1/2k+1}^{t-1/2}+\left|w_{ij-1/2k+1}^{t-1/2}\right|\right)u_{ijk}^{t}-\left(w_{ij-1/2k}^{t-1/2}+\left|w_{ij-1/2k}^{t-1/2}\right|\right)u_{ijk-1}^{t}}{H_{ij-1/2k}^{t}}+\frac{\left(w_{ij-1/2k+1}^{t-1/2}-\left|w_{ij-1/2k+1}^{t-1/2}\right|\right)u_{ijk+1}^{t}-\left(w_{ij-1/2k}^{t-1/2}-\left|w_{ij-1/2k}^{t-1/2}\right|\right)u_{ijk}^{t}}{H_{ij-1/2k}^{t}}\right]- \\
& -g\Delta t\left(\frac{\xi_{ij}^{t+1/2}-\xi_{ij-1}^{t+1/2}}{\Delta x_{ij-1/2k}}+\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{ij-1k}\right)\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)/2}{\left(\rho_{ijk}+\rho_{ij-1k}\right)/2\,\Delta x_{ij-1/2k}}\right) \\
& +\frac{\nu_{x}}{\Delta x_{ijk}^{2}}\left(u_{ij+1k}^{t}-u_{ijk}^{t}\right)\Delta t-\frac{\nu_{x}}{\Delta x_{ij-1k}^{2}}\left(u_{ijk}^{t}-u_{ij-1k}^{t}\right)\Delta t \\
& +\frac{\nu_{y}}{\Delta y_{i+1/2jk}^{2}}\left(u_{i+1jk}^{t}-u_{ijk}^{t}\right)\Delta t-\frac{\nu_{y}}{\Delta y_{i-1/2jk}^{2}}\left(u_{ijk}^{t}-u_{i-1jk}^{t}\right)\Delta t+f\,v_{ijk}^{*}
\end{aligned}
$$

(12)

$$
\begin{aligned}
& a_{ijk}u_{ijk-1}^{t+1}+b_{ijk}u_{ijk}^{t+1}+c_{ijk}u_{ijk+1}^{t+1}=r_{ijk} \\
& a_{ij}=0 \\
& b_{ijk}=\left(1+\frac{\Delta t}{H_{ijk}^{t}+H_{ij-1k}^{t}}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u_{ijk}^{t}\right|\right)+\frac{\Delta t\nu_{ijk+1}}{\Delta z_{ij-1/2k+1/2}^{2}} \qquad \textbf{(13)} \\
& c_{ijk}=-\frac{\Delta t\nu_{ijk+1}}{\Delta z_{ij-1/2k+1/2}^{2}}
\end{aligned}
$$

Upper layer:

At the upper layer it is necessary to consider the wind drag (last term on the right side of equation 14.

$$u_{ij}^{t+1}+\frac{\Delta t\nu_{ijk}}{\Delta z^2_{ij-1/2k-1/2}}\left(u_{ijk}^{t+1}-u_{ijk-1}^{t+1}\right)=$$

$$u_{ij}^{t}-\frac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left[\begin{array}{l}\dfrac{H_{ijk}^{t}\left(u_{ij+1/2k}^{t}+\left|u_{ij+1/2k}^{t}\right|\right)u_{ijk}^{t}-H_{ij-1k}^{t}\left(u_{ij-1/2k}^{t}+\left|u_{ij-1/2k}^{t}\right|\right)u_{ij-1k}^{t}}{\Delta x_{ij-1/2k}}+\\ \dfrac{H_{ijk}^{t}\left(u_{ij+1/2k}^{t}-\left|u_{ij+1/2k}^{t}\right|\right)u_{ij+1k}^{t}-H_{ij-1k}^{t}\left(u_{ij-1/2k}^{t}-\left|u_{ij-1/2k}^{t}\right|\right)u_{ijk}^{t}}{\Delta x_{ij-1/2k}}\end{array}\right]$$

$$-\frac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left[\begin{array}{l}\dfrac{H_{i+1/2j-1/2k}^{t}\left(v_{i+1/2j-1/2k}^{t}+\left|v_{i+1/2j-1/2k}^{t}\right|\right)u_{ijk}^{t}-H_{i-1/2j-1/2k}^{t}\left(v_{i-1/2j-1/2k}^{t}+\left|v_{i-1/2j-1/2k}^{t}\right|\right)u_{i-1jk}^{t}}{\Delta y_{i-1/2jk}}+\\ \dfrac{H_{i+1/2j-1/2k}^{t}\left(v_{i+1/2j-1/2k}^{t}-\left|v_{i+1/2j-1/2k}^{t}\right|\right)u_{i+1jk}^{t}-H_{i-1/2j-1/2k}^{t}\left(v_{i-1/2j-1/2k}^{t}-\left|v_{i-1/2j-1/2k}^{t}\right|\right)u_{ijk}^{t}}{\Delta y_{i-1/2jk}}\end{array}\right]$$

$$-\Delta t\left[\begin{array}{l}\dfrac{\left(w_{ij-1/2k+1}^{t-1/2}+\left|w_{ij-1/2k+1}^{t-1/2}\right|\right)u_{ijk}^{t}-\left(w_{ij-1/2k}^{t-1/2}+\left|w_{ij-1/2k}^{t-1/2}\right|\right)u_{ijk-1}^{t}}{2H_{ij-1/2k}^{t}}+\\ \dfrac{\left(w_{ij-1/2k+1}^{t-1/2}-\left|w_{ij-1/2k+1}^{t-1/2}\right|\right)u_{ijk+1}^{t}-\left(w_{ij-1/2k}^{t-1/2}-\left|w_{ij-1/2k}^{t-1/2}\right|\right)u_{ijk}^{t}}{2H_{ij-1/2k}^{t}}\end{array}\right]$$

$$--g\Delta t\left(\frac{\xi_{ij}^{t+1/2}-\xi_{ij-1}^{t+1/2}}{\Delta x_{ij-1/2k}}+\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{ij-1k}\right)\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)/2}{\left(\rho_{ijk}+\rho_{ij-1k}\right)/2\,\Delta x_{ij-1/2k}}\right)$$

$$+\frac{\nu_x}{\Delta x^2_{ijk}}\left(u_{ij+1k}^{t}-u_{ijk}^{t}\right)-\frac{\nu_x}{\Delta x^2_{ij-1k}}\left(u_{ijk}^{t}-u_{ij-1k}^{t}\right)$$

$$+\frac{\nu_y}{\Delta y^2_{i+1/2jk}}\left(u_{i+1jk}^{t}-u_{ijk}^{t}\right)-\frac{\nu_y}{\Delta y^2_{i-1/2jk}}\left(u_{ijk}^{t}-u_{i-1jk}^{t}\right)+\frac{\Delta t\tau_{ij}^{vento}}{\rho\Delta z_{ijk}}+f\,v_{ijk}^{*}$$

(14)

$$a_{ijk}u_{ijk-1}^{t+1}+b_{ijk}u_{ijk}^{t+1}+c_{ijk}u_{ijk+1}^{t+1}=r_{ijk}$$

$$a_{ijk}=-\frac{\Delta t\nu_{ijk}}{\Delta z^2_{ij-1/2k-1/2}}$$

(15)

$$b_{ijk}=1+\frac{\Delta t\nu_{ij}}{\Delta z^2_{ij-1/2k-1/2}}$$

$$c_{ijk}=0$$

3.1.2 South-North velocity component

In the following pages the North-South velocity component (referred as v) will be solved implicitly considering intermediate, bottom and surface layers. The explicit version of each equation will be presented first, followed by the implicit solution.

Intermediate layers:

$$
\begin{aligned}
&\frac{v_{ijk}^{t+1}-v_{ijk}^{t}}{\Delta t}+\\
&\frac{1}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left[\begin{array}{l}\dfrac{H_{ijk}^{t}\left(v_{i+1/2jk}^{t}+\left|v_{i+1/2jk}^{t}\right|\right)v_{ijk}^{t}-H_{i-1jk}^{t}\left(v_{i-1/2jk}^{t}+\left|v_{i-1/2jk}^{t}\right|\right)v_{i-1jk}^{t}}{\Delta y_{i-1/2jk}}+\\ \dfrac{H_{ijk}^{t}\left(v_{i+1/2jk}^{t}-\left|v_{i+1/2jk}^{t}\right|\right)v_{i+1jk}^{t}-H_{i-1jk}^{t}\left(v_{i-1/2jk}^{t}-\left|v_{i-1/2jk}^{t}\right|\right)v_{ijk}^{t}}{\Delta y_{i-1/2jk}}\end{array}\right]\\
&+\frac{1}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left[\begin{array}{l}\dfrac{H_{i-1/2j+1/2k}^{t}\left(u_{i-1/2j+1/2k}^{t}+\left|u_{i-1/2j+1/2k}^{t}\right|\right)v_{ijk}^{t}-H_{i-1/2j-1/2k}^{t}\left(u_{i-1/2j-1/2k}^{t}+\left|u_{i-1/2j-1/2k}^{t}\right|\right)v_{ij-1k}^{t}}{\Delta x_{ij-1/2k}}+\\ \dfrac{H_{i-1/2j+1/2k}^{t}\left(u_{i-1/2j+1/2k}^{t}-\left|u_{i-1/2j+1/2k}^{t}\right|\right)v_{ij+1k}^{t}-H_{i-1/2j-1/2k}^{t}\left(u_{i-1/2j-1/2k}^{t}-\left|u_{i-1/2j-1/2k}^{t}\right|\right)v_{ijk}^{t}}{\Delta x_{ij-1/2k}}\end{array}\right]\\
&+\left[\begin{array}{l}\dfrac{\left(w_{i-1/2jk+1}^{t-1/2}+\left|w_{i-1/2jk+1}^{t-1/2}\right|\right)v_{ijk}^{t}-\left(w_{i-1/2jk}^{t-1/2}+\left|w_{i-1/2jk}^{t-1/2}\right|\right)v_{ijk-1}^{t}}{2H_{i-1/2jk}^{t}}+\\ \dfrac{\left(w_{i-1/2jk+1}^{t-1/2}-\left|w_{i-1/2jk+1}^{t-1/2}\right|\right)v_{ijk+1}^{t}-\left(w_{i-1/2jk}^{t-1/2}-\left|w_{i-1/2jk}^{t-1/2}\right|\right)v_{ijk}^{t}}{2H_{i-1/2jk}^{t}}\end{array}\right]=\\
&-g\left(\frac{\xi_{ij}^{t+1/2}-\xi_{i-1j}^{t+1/2}}{\Delta y_{i-1/2jk}}+\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{i-1jk}\right)\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)/2}{\left(\rho_{ijk}+\rho_{i-1jk}\right)/2\,\Delta y_{i-1/2jk}}\right)\\
&+\frac{\nu_{x}}{\Delta y_{ijk}^{2}}\left(v_{i+1jk}^{t}-v_{ijk}^{t}\right)-\frac{\nu_{x}}{\Delta y_{i-1jk}^{2}}\left(v_{ijk}^{t}-v_{i-1jk}^{t}\right)\\
&+\frac{\nu_{y}}{\Delta x_{ij+1/2k}^{2}}\left(v_{ij+1k}^{t}-v_{ijk}^{t}\right)-\frac{\nu_{y}}{\Delta x_{ij-1/2k}^{2}}\left(v_{ijk}^{t}-v_{ij-1k}^{t}\right)\\
&+\frac{\nu_{ijk+1}}{\Delta z_{i-1/2jk+1/2}^{2}}\left(v_{ijk+1}^{t+1}-v_{ijk}^{t+1}\right)-\frac{\nu_{ijk}}{\Delta z_{i-1/2jk-1/2}^{2}}\left(v_{ijk}^{t+1}-v_{ijk-1}^{t+1}\right)-f\,u_{ijk}^{*}
\end{aligned}
$$

(16)

$$
-\frac{\Delta t \nu_{ijk}}{\Delta z^2_{i-1/2\,jk-1/2}} v^{t+1}_{ijk-1} + v^{t+1}_{ijk} + \frac{\Delta t \nu_{ijk}}{\Delta z^2_{i-1/2\,jk-1/2}} v^{t+1}_{ijk} + \frac{\Delta t \nu_{ijk+1}}{\Delta z^2_{i-1/2\,jk+1/2}} v^{t+1}_{ijk} - \frac{\Delta t \nu_{ijk+1}}{\Delta z^2_{i-1/2\,jk+1/2}} v^{t+1}_{ijk+1} = v^t_{ijk}
$$

$$
-\frac{\Delta t}{\left(H^t_{ijk}+H^t_{ij-1k}\right)}\left[\begin{array}{l}\dfrac{H^t_{ijk}\left(v^t_{i+1/2\,jk}+\left|v^t_{i+1/2\,jk}\right|\right)v^t_{ijk}-H^t_{i-1\,jk}\left(v^t_{i-1/2\,jk}+\left|v^t_{i-1/2\,jk}\right|\right)v^t_{i-1\,jk}}{\Delta y_{i-1/2\,jk}}+\\ \dfrac{H^t_{ijk}\left(v^t_{i+1/2\,jk}-\left|v^t_{i+1/2\,jk}\right|\right)v^t_{i+1\,jk}-H^t_{i-1\,jk}\left(v^t_{i-1/2\,jk}-\left|v^t_{i-1/2\,jk}\right|\right)v^t_{ijk}}{\Delta y_{i-1/2\,jk}}\end{array}\right]
$$

$$
-\frac{\Delta t}{\left(H^t_{ijk}+H^t_{ij-1k}\right)}\left[\begin{array}{l}\dfrac{H^t_{i-1/2\,j+1/2\,k}\left(u^t_{i-1/2\,j+1/2\,k}+\left|u^t_{i-1/2\,j+1/2\,k}\right|\right)v^t_{ijk}-H^t_{i-1/2\,j-1/2\,k}\left(u^t_{i-1/2\,j-1/2\,k}+\left|u^t_{i-1/2\,j-1/2\,k}\right|\right)v^t_{ij-1k}}{\Delta x_{ij-1/2\,k}}+\\ \dfrac{H^t_{i-1/2\,j+1/2\,k}\left(v^t_{i-1/2\,j+1/2\,k}-\left|v^t_{i-1/2\,j+1/2\,k}\right|\right)u^t_{ij+1k}-H^t_{i-1/2\,j-1/2\,k}\left(v^t_{i-1/2\,j-1/2\,k}-\left|v^t_{i-1/2\,j-1/2\,k}\right|\right)u^t_{ijk}}{\Delta x_{ij-1/2\,k}}\end{array}\right]
$$

$$
-\Delta t\left[\begin{array}{l}\dfrac{\left(w^{t-1/2}_{i-1/2\,jk+1}+\left|w^{t-1/2}_{i-1/2\,jk+1}\right|\right)v^t_{ijk}-\left(w^{t-1/2}_{i-1/2\,jk}+\left|w^{t-1/2}_{i-1/2\,jk}\right|\right)v^t_{ijk-1}}{2H^t_{i-1/2\,jk}}+\\ \dfrac{\left(w^{t-1/2}_{i-1/2\,jk+1}-\left|w^{t-1/2}_{i-1/2\,jk+1}\right|\right)v^t_{ijk+1}-\left(w^{t-1/2}_{i-1/2\,jk}-\left|w^{t-1/2}_{i-1/2\,jk}\right|\right)v^t_{ijk}}{2H^t_{i-1/2\,jk}}\end{array}\right]
$$

$$
-g\Delta t\left(\frac{\xi^{t+1/2}_{ij}-\xi^{t+1/2}_{i-1j}}{\Delta y_{ij-1/2}}+\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{kj}-\rho_{kj-1}\right)\left(H^t_{kj}+H^t_{kj-1}\right)/2}{\left(\rho_{ij}+\rho_{i-1j}\right)/2\,\Delta y_{i-1/2\,j}}\right)
$$

$$
+\frac{\nu_x}{\Delta y^2_{ijk}}\left(v^t_{i+1\,jk}-u^t_{ijk}\right)\Delta t-\frac{\nu_x}{\Delta y^2_{i-1\,jk}}\left(v^t_{ijk}-v^t_{i-1\,jk}\right)\Delta t
$$

$$
+\frac{\nu_y}{\Delta x^2_{ij+1/2\,k}}\left(v^t_{ij+1k}-v^t_{ijk}\right)\Delta t-\frac{\nu_y}{\Delta x^2_{ij-1/2\,k}}\left(v^t_{ijk}-v^t_{ij-1k}\right)\Delta t-f\,u^*_{ijk}
$$

(17)

$$
a_{jki}v^{t+1}_{ijk-1}+b_{jki}v^{t+1}_{ijk}+c_{jki}v^{t+1}_{ijk+1}=r_{ijk}
$$

$$
a_{jki}=-\frac{\Delta t \nu_{ijk}}{\Delta z^2_{i-1/2\,jk-1/2}}
$$

$$
b_{jki}=1+\frac{\Delta t \nu_{ijk}}{\Delta z^2_{i-1/2\,jk-1/2}}+\frac{\Delta t \nu_{ijk+1}}{\Delta z^2_{i-1/2\,jk+1/2}} \qquad \textbf{(18)} \qquad \begin{vmatrix} b_{jki} & c_{jki} & \cdots & \cdots \\ a_{jki} & b_{jki} & c_{jki} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & a_{jki+n} & b_{jki+n} \end{vmatrix} \begin{vmatrix} v^{t+1}_{ijk} \\ v^{t+1}_{ijk+1} \\ \cdots \\ \ldots \end{vmatrix} = \begin{vmatrix} r_{ijk} \\ \cdots \\ \cdots \\ \cdots \end{vmatrix}
$$

$$
c_{jki}=-\frac{\Delta t \nu_{ijk+1}}{\Delta z^2_{i-1/2\,jk+1/2}}
$$

Bottom layer:

At the bottom layer it is necessary to consider bottom drag (last term on the right side of equation 19.

$$
\begin{aligned}
&\frac{v_{ijk}^{t+1}-v_{ijk}^{t}}{\Delta t} \\
&+\frac{1}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left[\begin{array}{l}
\dfrac{H_{ijk}^{t}\left(v_{i+\frac{1}{2}jk}^{t}+\left|v_{i+\frac{1}{2}jk}^{t}\right|\right)v_{ijk}^{t}-H_{i-1jk}^{t}\left(v_{i-\frac{1}{2}jk}^{t}+\left|v_{i-\frac{1}{2}jk}^{t}\right|\right)v_{i-1jk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}+ \\
\dfrac{H_{ijk}^{t}\left(v_{i+\frac{1}{2}jk}^{t}-\left|v_{i+\frac{1}{2}jk}^{t}\right|\right)v_{i+1jk}^{t}-H_{i-1jk}^{t}\left(v_{i-\frac{1}{2}jk}^{t}-\left|v_{i-\frac{1}{2}jk}^{t}\right|\right)v_{ijk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}
\end{array}\right] \\
&+\frac{1}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left[\begin{array}{l}
\dfrac{H_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}\left(u_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}+\left|u_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}\right|\right)v_{ijk}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(u_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}+\left|u_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)v_{ij-1k}^{t}}{\Delta x_{ij-\frac{1}{2}k}}+ \\
\dfrac{H_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}\left(u_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}-\left|u_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}\right|\right)v_{ij+1k}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(u_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}-\left|u_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)v_{ijk}^{t}}{\Delta x_{ij-\frac{1}{2}k}}
\end{array}\right] \\
&+\left[\begin{array}{l}
\dfrac{\left(w_{i-\frac{1}{2}jk+1}^{t-\frac{1}{2}}+\left|w_{i-\frac{1}{2}jk+1}^{t-\frac{1}{2}}\right|\right)v_{ijk}^{t}-\left(w_{i-\frac{1}{2}jk}^{t-\frac{1}{2}}+\left|w_{i-\frac{1}{2}jk}^{t-\frac{1}{2}}\right|\right)v_{ijk-1}^{t}}{2H_{i-\frac{1}{2}jk}^{t}}+ \\
\dfrac{\left(w_{i-\frac{1}{2}jk+1}^{t-\frac{1}{2}}-\left|w_{i-\frac{1}{2}jk+1}^{t-\frac{1}{2}}\right|\right)v_{ijk+1}^{t}-\left(w_{i-\frac{1}{2}jk}^{t-\frac{1}{2}}-\left|w_{i-\frac{1}{2}jk}^{t-\frac{1}{2}}\right|\right)v_{ijk}^{t}}{2H_{i-\frac{1}{2}jk}^{t}}
\end{array}\right]= \\
&-g\left(\frac{\xi_{ij}^{t+\frac{1}{2}}-\xi_{i-1j}^{t+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}jk}}+\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{i-1jk}\right)\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)/2}{\left(\rho_{ijk}+\rho_{i-1jk}\right)/2\,\Delta y_{i-\frac{1}{2}jk}}\right) \\
&+\frac{\nu_{x}}{\Delta y_{ijk}^{2}}\left(v_{i+1jk}^{t}-v_{ijk}^{t}\right)-\frac{\nu_{x}}{\Delta y_{i-1jk}^{2}}\left(v_{ijk}^{t}-v_{i-1jk}^{t}\right) \\
&+\frac{\nu_{y}}{\Delta x_{ij+\frac{1}{2}k}^{2}}\left(v_{ij+1k}^{t}-v_{ijk}^{t}\right)-\frac{\nu_{y}}{\Delta x_{ij-\frac{1}{2}k}^{2}}\left(v_{ijk}^{t}-v_{ij-1k}^{t}\right) \\
&+\frac{\nu_{ijk+1}}{\Delta z_{i-\frac{1}{2}jk+\frac{1}{2}}^{2}}\left(v_{ijk+1}^{t+1}-v_{ijk}^{t+1}\right) \\
&-\frac{1}{H_{ijk}^{t}+H_{i-1jk}^{t}}\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^{t}\right|v_{ijk}^{t+1}-f\,u_{ijk}^{*}
\end{aligned}
$$

(19)

Again, this equation is rearranged in order to obtain a tridiagonal system (20 and 21). However, due to stability criteria (see above) the bottom drag term is solved implicitly, reason why it is on the left side of equation 20.

$$
\begin{aligned}
& v_{ijk}^{t+1}\left(1+\frac{\Delta t}{H_{ijk}^{t}+H_{i-1jk}^{t}}\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^{t}\right|\right)+v_{ijk}^{t+1}\frac{\Delta t\nu_{ijk+1}}{\Delta z_{i-1/2jk+1/2}^{2}}-v_{i+1j}^{t+1}\frac{\Delta t\nu_{ijk+1}}{\Delta z_{i-1/2jk+1/2}^{2}}= \\
& u_{ijk}^{t}-\frac{\Delta t}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left[\begin{array}{l}\dfrac{H_{ijk}^{t}\left(v_{i+1/2jk}^{t}+\left|v_{i+1/2jk}^{t}\right|\right)v_{ijk}^{t}-H_{i-1jk}^{t}\left(v_{i-1/2jk}^{t}+\left|v_{i-1/2jk}^{t}\right|\right)v_{i-1jk}^{t}}{\Delta y_{i-1/2jk}}+\\ \dfrac{H_{ijk}^{t}\left(v_{i+1/2jk}^{t}-\left|v_{i+1/2jk}^{t}\right|\right)v_{i+1jk}^{t}-H_{i-1jk}^{t}\left(uv_{i-1/2jk}^{t}-\left|v_{i-1/2jk}^{t}\right|\right)v_{ijk}^{t}}{\Delta y_{i-1/2jk}}\end{array}\right] \\
& -\frac{\Delta t}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left[\begin{array}{l}\dfrac{H_{i-1/2j+1/2k}^{t}\left(u_{i-1/2j+1/2k}^{t}+\left|u_{i-1/2j+1/2k}^{t}\right|\right)v_{ijk}^{t}-H_{i-1/2j-1/2k}^{t}\left(u_{i-1/2j-1/2k}^{t}+\left|u_{i-1/2j-1/2k}^{t}\right|\right)v_{ij-1k}^{t}}{\Delta x_{ij-1/2k}}+\\ \dfrac{H_{i-1/2j+1/2k}^{t}\left(u_{i-1/2j+1/2k}^{t}-\left|u_{i-1/2j+1/2k}^{t}\right|\right)v_{ij+1k}^{t}-H_{i-1/2j-1/2k}^{t}\left(u_{i-1/2j-1/2k}^{t}-\left|u_{i-1/2j-1/2k}^{t}\right|\right)v_{ijk}^{t}}{\Delta x_{ij-1/2k}}\end{array}\right] \\
& -\Delta t\left[\begin{array}{l}\dfrac{\left(w_{i-1/2jk+1}^{t-1/2}+\left|w_{i-1/2jk+1}^{t-1/2}\right|\right)v_{ijk}^{t}-\left(w_{i-1/2jk}^{t-1/2}+\left|w_{i-1/2jk}^{t-1/2}\right|\right)v_{ijk-1}^{t}}{H_{i-1/2jk}^{t}}+\\ \dfrac{\left(w_{i-1/2jk+1}^{t-1/2}-\left|w_{i-1/2jk+1}^{t-1/2}\right|\right)v_{ijk+1}^{t}-\left(w_{i-1/2jk}^{t-1/2}-\left|w_{i-1/2jk}^{t-1/2}\right|\right)v_{ijk}^{t}}{H_{i-1/2jk}^{t}}\end{array}\right]- \\
& -g\Delta t\left(\frac{\xi_{ij}^{t+1/2}-\xi_{i-1j}^{t+1/2}}{\Delta y_{i-1/2jk}}+\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{i-1jk}\right)\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)/2}{\left(\rho_{ijk}+\rho_{i-1jk}\right)/2\,\Delta y_{i-1/2jk}}\right) \\
& +\frac{\nu_{x}}{\Delta y_{ijk}^{2}}\left(v_{i+1jk}^{t}-v_{ijk}^{t}\right)\Delta t-\frac{\nu_{x}}{\Delta y_{i-1jk}^{2}}\left(v_{ijk}^{t}-v_{i-1jk}^{t}\right)\Delta t \\
& +\frac{\nu_{y}}{\Delta x_{ij+1/2k}^{2}}\left(v_{ij+1\pi k}^{t}-v_{ijk}^{t}\right)\Delta t-\frac{\nu_{y}}{\Delta x_{ij-1/2k}^{2}}\left(v_{ijk}^{t}-v_{ij-1k}^{t}\right)\Delta t-f\,u_{ijk}^{*}
\end{aligned}
$$

(20)

$$
\begin{aligned}
& a_{ijk}v_{ijk-1}^{t+1}+b_{ijk}v_{ijk}^{t+1}+c_{ijk}v_{ijk+1}^{t+1}=r_{ijk} \\
& a_{ij}=0 \\
& b_{ijk}=\left(1+\frac{\Delta t}{H_{ijk}^{t}+H_{i-1jk}^{t}}\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^{t}\right|\right)+\frac{\Delta t\nu_{ijk+1}}{\Delta z_{i-1/2jk+1/2}^{2}} \qquad \textbf{(21)} \\
& c_{ijk}=-\frac{\Delta t\nu_{ijk+1}}{\Delta z_{i-1/2jk+1/2}^{2}}
\end{aligned}
$$

Upper layer:

At the upper layer it is necessary to consider the wind drag (last term on the right side of equation 22.

$$
\begin{aligned}
&\frac{v_{ijk}^{t+1}-v_{ijk}^{t}}{\Delta t}\\
&+\frac{1}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left[\begin{array}{l}\dfrac{H_{ijk}^{t}\left(v_{i+1/2jk}^{t}+\left|v_{i+1/2jk}^{t}\right|\right)v_{ijk}^{t}-H_{i-1jk}^{t}\left(v_{i-1/2jk}^{t}+\left|v_{i-1/2jk}^{t}\right|\right)v_{i-1jk}^{t}}{\Delta y_{i-1/2jk}}+\\ \dfrac{H_{ijk}^{t}\left(v_{i+1/2jk}^{t}-\left|v_{i+1/2jk}^{t}\right|\right)v_{i+1jk}^{t}-H_{i-1jk}^{t}\left(v_{i-1/2jk}^{t}-\left|v_{i-1/2jk}^{t}\right|\right)v_{ijk}^{t}}{\Delta y_{i-1/2jk}}\end{array}\right]\\
&+\frac{1}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left[\begin{array}{l}\dfrac{H_{i-1/2j+1/2k}^{t}\left(u_{i-1/2j+1/2k}^{t}+\left|u_{i-1/2j+1/2k}^{t}\right|\right)v_{ijk}^{t}-H_{i-1/2j-1/2k}^{t}\left(u_{i-1/2j-1/2k}^{t}+\left|u_{i-1/2j-1/2k}^{t}\right|\right)v_{ij-1k}^{t}}{\Delta x_{ij-1/2k}}+\\ \dfrac{H_{i-1/2j+1/2k}^{t}\left(u_{i-1/2j+1/2k}^{t}-\left|u_{i-1/2j+1/2k}^{t}\right|\right)v_{i+1jk}^{t}-H_{i-1/2j-1/2k}^{t}\left(u_{i-1/2j-1/2k}^{t}-\left|u_{i-1/2j-1/2k}^{t}\right|\right)v_{ijk}^{t}}{\Delta x_{ij-1/2k}}\end{array}\right]\\
&+\left[\begin{array}{l}\dfrac{\left(w_{i-1/2jk+1}^{t-1/2}+\left|w_{i-1/2jk+1}^{t-1/2}\right|\right)v_{ijk}^{t}-\left(w_{i-1/2jk}^{t-1/2}+\left|w_{i-1/2jk}^{t-1/2}\right|\right)v_{ijk-1}^{t}}{2H_{i-1/2jk}^{t}}+\\ \dfrac{\left(w_{i-1/2jk+1}^{t-1/2}-\left|w_{i-1/2jk+1}^{t-1/2}\right|\right)v_{ijk+1}^{t}-\left(w_{i-1/2jk}^{t-1/2}-\left|w_{i-1/2jk}^{t-1/2}\right|\right)v_{ijk}^{t}}{2H_{i-1/2jk}^{t}}\end{array}\right]=\\
&-g\left(\frac{\xi_{ij}^{t+1/2}-\xi_{i-1j}^{t+1/2}}{\Delta y_{ij-1/2k}}+\frac{\sum\limits_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{i-1jk}\right)\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)/2}{\left(\rho_{ijk}+\rho_{i-1jk}\right)/2\,\Delta y_{i-1/2jk}}\right)\\
&+\frac{\nu_{x}}{\Delta y_{ijk}^{2}}\left(v_{i+1jk}^{t}-v_{ijk}^{t}\right)-\frac{\nu_{x}}{\Delta y_{i-1jk}^{2}}\left(v_{ijk}^{t}-v_{i-1jk}^{t}\right)\\
&+\frac{\nu_{y}}{\Delta x_{ij+1/2k}^{2}}\left(v_{ij+1k}^{t}-v_{ijk}^{t}\right)-\frac{\nu_{y}}{\Delta x_{ij-1/2k}^{2}}\left(v_{ijk}^{t}-v_{ij-1k}^{t}\right)\\
&-\frac{\nu_{ijk}}{\Delta z_{i-1/2jk-1/2}^{2}}\left(v_{ijk}^{t+1}-v_{ijk-1}^{t+1}\right)+\frac{\tau_{ij}^{vento}}{\rho\Delta z_{ijk}}-f\,u_{ijk}^{*}
\end{aligned}
$$

(22)

$$
\begin{aligned}
& v_{ij}^{t+1}+\frac{\Delta t\nu_{ijk}}{\Delta z^2_{i-\frac{1}{2}jk-\frac{1}{2}}}\left(v_{ij}^{t+1}-v_{ij-1}^{t+1}\right)= \\
& v_{ij}^{t}-\frac{\Delta t}{\left(H^t_{ijk}+H^t_{i-1jk}\right)}\left[\begin{array}{l}\dfrac{H^t_{ijk}\left(v^t_{i+\frac{1}{2}jk}+\left|v^t_{i+\frac{1}{2}jk}\right|\right)v^t_{ijk}-H^t_{i-1jk}\left(v^t_{i-\frac{1}{2}jk}+\left|v^t_{i-\frac{1}{2}jk}\right|\right)v^t_{i-1jk}}{\Delta y_{i-\frac{1}{2}jk}}+ \\ \dfrac{H^t_{ijk}\left(v^t_{i+\frac{1}{2}jk}-\left|v^t_{i+\frac{1}{2}jk}\right|\right)v^t_{i+1jk}-H^t_{i-1jk}\left(v^t_{i-\frac{1}{2}jk}-\left|v^t_{i-\frac{1}{2}jk}\right|\right)v^t_{ijk}}{\Delta y_{i-\frac{1}{2}jk}}\end{array}\right] \\
& -\frac{\Delta t}{\left(H^t_{ijk}+H^t_{i-1jk}\right)}\left[\begin{array}{l}\dfrac{H^t_{i-\frac{1}{2}j+\frac{1}{2}k}\left(u^t_{i-\frac{1}{2}j+\frac{1}{2}k}+\left|u^t_{i-\frac{1}{2}j+\frac{1}{2}k}\right|\right)v^t_{ijk}-H^t_{i-\frac{1}{2}j-\frac{1}{2}k}\left(u^t_{i-\frac{1}{2}j-\frac{1}{2}k}+\left|u^t_{i-\frac{1}{2}j-\frac{1}{2}k}\right|\right)v^t_{ij-1k}}{\Delta x_{ij-\frac{1}{2}k}}+ \\ \dfrac{H^t_{i-\frac{1}{2}j+\frac{1}{2}k}\left(u^t_{i-\frac{1}{2}j+\frac{1}{2}k}-\left|u^t_{i-\frac{1}{2}j+\frac{1}{2}k}\right|\right)v^t_{ij+1k}-H^t_{i-\frac{1}{2}j-\frac{1}{2}k}\left(u^t_{i-\frac{1}{2}j-\frac{1}{2}k}-\left|u^t_{i-\frac{1}{2}j-\frac{1}{2}k}\right|\right)v^t_{ijk}}{\Delta x_{ij-\frac{1}{2}k}}\end{array}\right] \\
& -\Delta t\left[\begin{array}{l}\dfrac{\left(w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk+1}+\left|w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk+1}\right|\right)v^t_{ijk}-\left(w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk}+\left|w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk}\right|\right)v^t_{ijk-1}}{2H^t_{i-\frac{1}{2}jk}}+ \\ \dfrac{\left(w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk+1}-\left|w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk+1}\right|\right)v^t_{ijk+1}-\left(w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk}-\left|w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk}\right|\right)v^t_{ijk}}{2H^t_{i-\frac{1}{2}jk}}\end{array}\right] \\
& --g\Delta t\left(\frac{\xi^{t+\frac{1}{2}}_{ij}-\xi^{t+\frac{1}{2}}_{i-1j}}{\Delta y_{ij-\frac{1}{2}k}}+\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{i-1jk}\right)\left(H^t_{ijk}+H^t_{i-1jk}\right)/2}{\left(\rho_{ijk}+\rho_{i-1jk}\right)/2\,\Delta y_{i-\frac{1}{2}jk}}\right) \\
& +\frac{\nu_x}{\Delta y^2_{ijk}}\left(v^t_{i+1jk}-v^t_{ijk}\right)\Delta t-\frac{\nu_x}{\Delta y^2_{i-1jk}}\left(v^t_{ijk}-v^t_{i-1jk}\right)\Delta t \\
& +\frac{\nu_y}{\Delta x^2_{ij+\frac{1}{2}k}}\left(v^t_{ij+1k}-v^t_{ijk}\right)\Delta t-\frac{\nu_y}{\Delta x^2_{ij-\frac{1}{2}k}}\left(v^t_{ijk}-v^t_{ij-1k}\right)\Delta t+\frac{\Delta t\tau^{vento}_{ij}}{\rho\Delta z_{ijk}}-f\,u^*_{ijk}\Delta t
\end{aligned}
$$

(23)

$$
\begin{aligned}
& a_{ijk}v^{t+1}_{ijk-1}+b_{ijk}v^{t+1}_{ijk}+c_{ijk}v^{t+1}_{ijk+1}=r_{ijk} \\
& a_{ijk}=-\frac{\Delta t\nu_{ijk}}{\Delta z^2_{i-\frac{1}{2}jk-\frac{1}{2}}} \\
& b_{ijk}=1+\frac{\Delta t\nu_{ij}}{\Delta z^2_{i-\frac{1}{2}jk-\frac{1}{2}}} \\
& c_{ijk}=0
\end{aligned}
\qquad \mathbf{(24)}
$$

Again, this equation is rearranged in order to obtain a tridiagonal system (23 and 24). The three different solutions of the terms *a*, *b*, *c* and *r* (equations 18, 21 and 24) are applied according to the type of layers.

3.2 Equation of continuity

3.2.1 Water level estimation

The numerical version of the equation of continuity (3) is presented in equation 25. The elevation is calculated from the difference between the total inflows and the total outflows of each grid column, making unnecessary to know the values of vertical velocities at this point. The terms "*FlowInput*" and "*FlowOutput*" are included to account for any water discharges and uptakes.

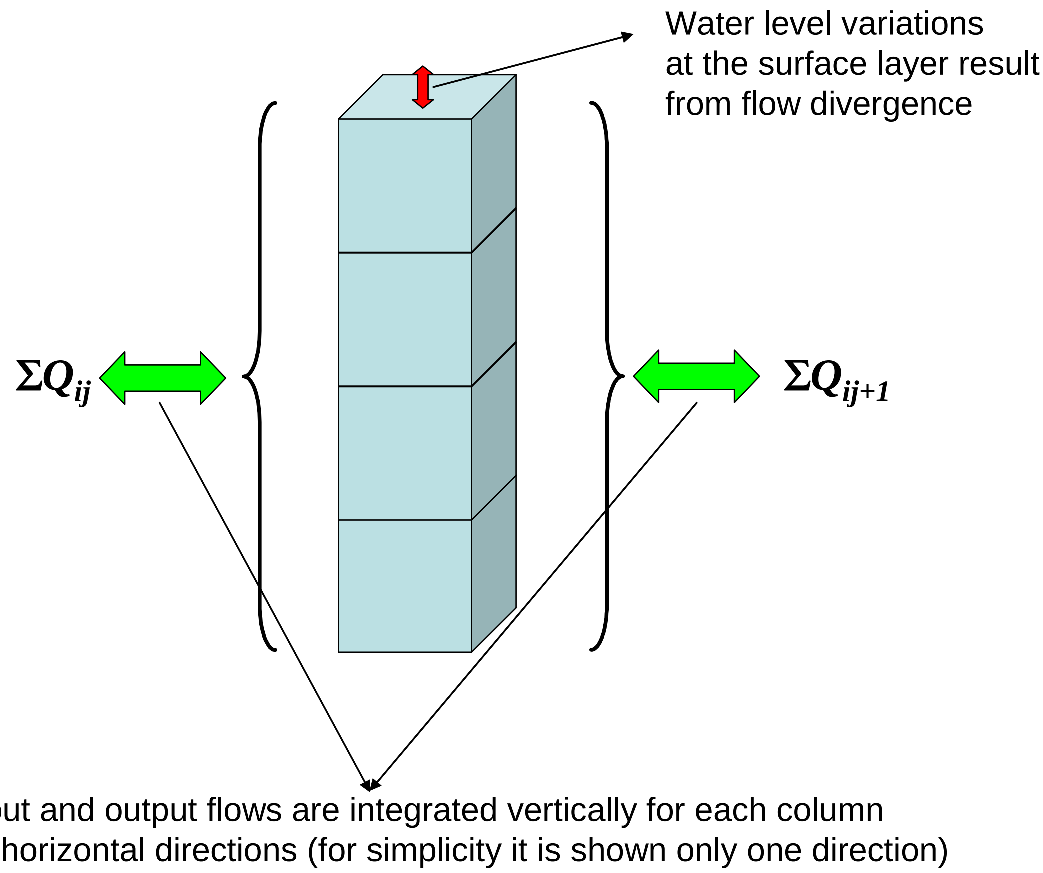

Figure 2 – A columns of cells, illustrating how the equation of continuity is used to estimate water level to be used in the barotropic term of the u and v equations.

Semi time step = 1

$$
\begin{aligned}
\xi_{ij}^{t+1/2} = \xi_{ij}^{t} &+ \Delta t \sum_{k=bottom}^{surface} \left(\frac{u_{ijk}^{t+1}\left(H_{ijk}^{t} + H_{ij-1k}^{t}\right)}{2\Delta x_{ijk}} - \frac{u_{ij+1k}^{t+1}\left(H_{ijk}^{t} + H_{ij+1k}^{t}\right)}{2\Delta x_{ijk}} \right) \\
&+ \Delta t \sum_{k=bottom}^{surface} \left(\frac{v_{ijk}^{t}\left(H_{ijk}^{t} + H_{i-1jk}^{t}\right)}{2\Delta y_{ijk}} - \frac{v_{i+1jk}^{t}\left(H_{ijk}^{t} + H_{i+1jk}^{t}\right)}{2\Delta y_{ijk}} \right) \\
&+ \Delta t \sum_{k=bottom}^{surface} \left(\frac{FlowInput_{ijk} - FlowOutput_{ijk}}{\Delta x_{ijk} \Delta y_{ijk}} \right)
\end{aligned}
\qquad \textbf{(25)}
$$

The velocity terms in *t+1* may be obtained by solving equations 8, 11 and 14 for u_{ij}^{t+1} and summing them over each column obtaining equation 26. The vertical diffusion terms cancel each other and it is possible to obtain an equation for u^{t+1} to use in equation 25.

$$
\begin{aligned}
&\sum_{k=bottom}^{surface}\left[u_{ijk}^{t+1}\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\right]=\sum_{k=bottom}^{surface}\left[u_{ij}^{t}\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\right]\\
&-\sum_{k=bottom}^{surface}\left[\begin{aligned}
&\Delta t\left[\begin{aligned}&\frac{H_{ijk}^{t}\left(u_{ij+\frac{1}{2}k}^{t}+\left|u_{ij+\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}-H_{ij-1k}^{t}\left(u_{ij-\frac{1}{2}k}^{t}+\left|u_{ij-\frac{1}{2}k}^{t}\right|\right)u_{ij-1k}^{t}}{\Delta x_{ij-\frac{1}{2}k}}+\\
&\frac{H_{ijk}^{t}\left(u_{ij+\frac{1}{2}k}^{t}-\left|u_{ij+\frac{1}{2}k}^{t}\right|\right)u_{ij+1k}^{t}-H_{ij-1k}^{t}\left(u_{ij-\frac{1}{2}k}^{t}-\left|u_{ij-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}}{\Delta x_{ij-\frac{1}{2}k}}\end{aligned}\right]\\
&+\Delta t\left[\begin{aligned}&\frac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}+\left|v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}+\left|v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{i-1jk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}+\\
&\frac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}-\left|v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{i+1jk}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}-\left|v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}\end{aligned}\right]\\
&+\Delta t\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\\
&\left[\begin{aligned}&\frac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}+\left|w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}\right|\right)u_{ijk}^{t}-\left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}+\left|w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}\right|\right)u_{ijk-1}^{t}}{2H_{ij-\frac{1}{2}k}^{t}}+\\
&\frac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}-\left|w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}\right|\right)u_{ijk+1}^{t}-\left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}-\left|w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}\right|\right)u_{ijk}^{t}}{2H_{ij-\frac{1}{2}k}^{t}}\end{aligned}\right]
\end{aligned}\right]\\
&-g\Delta t\sum_{k=bottom}^{surface}\left(\begin{aligned}&\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\frac{\xi_{ij}^{t+\frac{1}{2}}-\xi_{ij-1}^{t+\frac{1}{2}}}{\Delta x_{ij-\frac{1}{2}k}}\\
&+\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{ij-1k}\right)\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)/2}{\left(\rho_{ijk}+\rho_{ij-1k}\right)/2\,\Delta x_{ij-\frac{1}{2}k}}\end{aligned}\right)\\
&+\sum_{k=bottom}^{surface}\left[\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\Delta t\left(\frac{\nu_{x}}{\Delta x_{ijk}^{2}}\left(u_{ij+1k}^{t}-u_{ijk}^{t}\right)-\frac{\nu_{x}}{\Delta x_{ij-1k}^{2}}\left(u_{ijk}^{t}-u_{ij-1k}^{t}\right)\right)\right]\\
&+\sum_{k=bottom}^{surface}\left[\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\Delta t\left(\frac{\nu_{y}}{\Delta y_{i+\frac{1}{2}jk}^{2}}\left(u_{i+1jk}^{t}-u_{ijk}^{t}\right)-\frac{\nu_{y}}{\Delta y_{i-\frac{1}{2}jk}^{2}}\left(u_{ijk}^{t}-u_{i-1jk}^{t}\right)\right)\right]\\
&+\frac{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\Delta t\,\tau_{ij}^{vento}}{\rho H_{ij-\frac{1}{2}k}^{t}}-\Delta t\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u_{ijk}^{t}\right|u_{ijk}^{t+1}+\Delta t\sum_{k=bottom}^{surface}f\,v_{ijk}^{*}
\end{aligned}
$$

(26)

However, there is still a velocity in *t+1* on the right side of equation 26, in the bottom drag term. This term may be replaced by solving equation 11 for u_{ij}^{t+1} as shown in equations 27 and 28. Equation 29 is just a simplified version of 28.

$$
\begin{aligned}
&u_{ijk}^{t+1}\left(1+\frac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u_{ijk}^{t}\right|\right)=u_{ijk}^{t}\\
&-\frac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left[\begin{array}{l}\dfrac{H_{ijk}^{t}\left(u_{ij+\frac{1}{2}k}^{t}+\left|u_{ij+\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}-H_{ij-1k}^{t}\left(u_{ij-\frac{1}{2}k}^{t}+\left|u_{ij-\frac{1}{2}k}^{t}\right|\right)u_{ij-1k}^{t}}{\Delta x_{ij-\frac{1}{2}k}}+\\ \dfrac{H_{ijk}^{t}\left(u_{ij+\frac{1}{2}k}^{t}-\left|u_{ij+\frac{1}{2}k}^{t}\right|\right)u_{ij+1k}^{t}-H_{ij-1k}^{t}\left(u_{ij-\frac{1}{2}k}^{t}-\left|u_{ij-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}}{\Delta x_{ij-\frac{1}{2}k}}\end{array}\right]\\
&-\frac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left[\begin{array}{l}\dfrac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}+\left|v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}+\left|v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{i-1jk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}+\\ \dfrac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}-\left|v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{i+1jk}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}-\left|v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}\end{array}\right]\\
&-\Delta t\left[\begin{array}{l}\dfrac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}+\left|w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}\right|\right)u_{ijk}^{t}-\left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}+\left|w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}\right|\right)u_{ijk-1}^{t}}{2H_{ij-\frac{1}{2}k}^{t}}+\\ \dfrac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}-\left|w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}\right|\right)u_{ijk+1}^{t}-\left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}-\left|w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}\right|\right)u_{ijk}^{t}}{2H_{ij-\frac{1}{2}k}^{t}}\end{array}\right]\\
&-g\Delta t\left(\frac{\xi_{ij}^{t+\frac{1}{2}}-\xi_{ij-1}^{t+\frac{1}{2}}}{\Delta x_{ij-\frac{1}{2}k}}+\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{ij-1k}\right)\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)/2}{\left(\rho_{ijk}+\rho_{ij-1k}\right)/2\,\Delta x_{ij-\frac{1}{2}k}}\right)\\
&+\frac{\nu_{x}}{\Delta x_{ijk}^{2}}\left(u_{ij+1k}^{t}-u_{ijk}^{t}\right)\Delta t-\frac{\nu_{x}}{\Delta x_{ij-1k}^{2}}\left(u_{ijk}^{t}-u_{ij-1k}^{t}\right)\Delta t\\
&+\frac{\nu_{y}}{\Delta y_{i+\frac{1}{2}jk}^{2}}\left(u_{i+1jk}^{t}-u_{ijk}^{t}\right)\Delta t-\frac{\nu_{y}}{\Delta y_{i-\frac{1}{2}jk}^{2}}\left(u_{ijk}^{t}-u_{i-1jk}^{t}\right)\Delta t\\
&+\frac{\nu_{ijk+1}}{\Delta z_{ij-\frac{1}{2}k+\frac{1}{2}}^{2}}\left(u_{ijk+1}^{t+1}-u_{ijk}^{t+1}\right)\Delta t+f\,v_{ijk}^{*}\Delta t
\end{aligned}
$$

(27)

$$
u_{ijk}^{t+1}=\frac{\left[\begin{aligned}
&u_{ijk}^{t}-\frac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left[\begin{aligned}
&\frac{H_{ijk}^{t}\left(u_{ij+\frac{1}{2}k}^{t}+\left|u_{ij+\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}-H_{ij-1k}^{t}\left(u_{ij-\frac{1}{2}k}^{t}+\left|u_{ij-\frac{1}{2}k}^{t}\right|\right)u_{ij-1k}^{t}}{\Delta x_{ij-\frac{1}{2}k}}+\\
&\frac{H_{ijk}^{t}\left(u_{ij+\frac{1}{2}k}^{t}-\left|u_{ij+\frac{1}{2}k}^{t}\right|\right)u_{ij+1k}^{t}-H_{ij-1k}^{t}\left(u_{ij-\frac{1}{2}k}^{t}-\left|u_{ij-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}}{\Delta x_{ij-\frac{1}{2}k}}
\end{aligned}\right]\\
&-\frac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left[\begin{aligned}
&\frac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}+\left|v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}+\left|v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{i-1jk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}+\\
&\frac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}-\left|v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{i+1jk}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}-\left|v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)u_{ijk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}
\end{aligned}\right]\\
&-\Delta t\left[\begin{aligned}
&\frac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}+\left|w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}\right|\right)u_{ijk}^{t}-\left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}+\left|w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}\right|\right)u_{ijk-1}^{t}}{2H_{ij-\frac{1}{2}k}^{t}}+\\
&\frac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}-\left|w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}\right|\right)u_{ijk+1}^{t}-\left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}-\left|w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}\right|\right)u_{ijk}^{t}}{2H_{ij-\frac{1}{2}k}^{t}}
\end{aligned}\right]\\
&-g\Delta t\left(\frac{\xi_{ij}^{t+\frac{1}{2}}-\xi_{ij-1}^{t+\frac{1}{2}}}{\Delta x_{ij-\frac{1}{2}k}}+\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{ij-1k}\right)\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)/2}{\left(\rho_{ijk}+\rho_{ij-1k}\right)/2\,\Delta x_{ij-\frac{1}{2}k}}\right)\\
&+\frac{\nu_x}{\Delta x_{ijk}^{2}}\left(u_{ij+1k}^{t}-u_{ijk}^{t}\right)\Delta t-\frac{\nu_x}{\Delta x_{ij-1k}^{2}}\left(u_{ijk}^{t}-u_{ij-1k}^{t}\right)\Delta t\\
&+\frac{\nu_y}{\Delta y_{i+\frac{1}{2}jk}^{2}}\left(u_{i+1jk}^{t}-u_{ijk}^{t}\right)\Delta t-\frac{\nu_y}{\Delta y_{i-\frac{1}{2}jk}^{2}}\left(u_{ijk}^{t}-u_{i-1jk}^{t}\right)\Delta t\\
&+\frac{\nu_{ijk+1}}{\Delta z_{ij-\frac{1}{2}k+\frac{1}{2}}^{2}}\left(u_{ijk+1}^{t+1}-u_{ijk}^{t+1}\right)\Delta t+f\,v_{ijk}^{*}\Delta t
\end{aligned}\right]}{\left(1+\frac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u_{ijk}^{t}\right|\right)}
$$

(28)

$$
u_{ijk}^{t+1}=\frac{\left[-g\Delta t\left(\frac{\xi_{ij}^{t+\frac{1}{2}}-\xi_{ij-1}^{t+\frac{1}{2}}}{\Delta x_{ij-\frac{1}{2}k}}\right)+Other\_terms\right]}{\left(1+\frac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u_{ijk}^{t}\right|\right)}\quad (29)
$$

Equation 30 is obtained by inserting equation 27 into equation 26.

$$
\begin{aligned}
&\xi_{ij}^{t+\frac{1}{2}} + \Delta t g \Delta t \sum_{k=bottom}^{surface} \left(\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \frac{\xi_{ij}^{t+\frac{1}{2}} - \xi_{ij-1}^{t+\frac{1}{2}}}{\Delta x_{ij-\frac{1}{2}}} \Big/ \left(2\Delta x_{ij} \right) \right) \\
&- \Delta t g \Delta t \sum_{k=bottom}^{surface} \left(\left(H_{ij+1k}^{t} + H_{ijk}^{t} \right) \frac{\xi_{ij+1}^{t+\frac{1}{2}} - \xi_{ij}^{t+\frac{1}{2}}}{\Delta x_{ij+\frac{1}{2}}} \Big/ \left(2\Delta x_{ij} \right) \right) = \xi_{ij}^{t} + \Delta t \sum_{k=bottom}^{surface} \left[u_{ijk}^{t} \left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) / 2\Delta x_{ij} \right] \\
&- \Delta t \sum_{k=bottom}^{surface} \left[\begin{array}{l}
\Delta t \left[\begin{array}{l}
\dfrac{H_{ijk}^{t} \left(u_{ij+\frac{1}{2}k}^{t} + \left| u_{ij+\frac{1}{2}k}^{t} \right| \right) u_{ijk}^{t} - H_{ij-1k}^{t} \left(u_{ij-\frac{1}{2}k}^{t} + \left| u_{ij-\frac{1}{2}k}^{t} \right| \right) u_{ij-1k}^{t}}{\Delta x_{ij-\frac{1}{2}k}} + \\
\dfrac{H_{ijk}^{t} \left(u_{ij+\frac{1}{2}k}^{t} - \left| u_{ij+\frac{1}{2}k}^{t} \right| \right) u_{ij+1k}^{t} - H_{ij-1k}^{t} \left(u_{ij-\frac{1}{2}k}^{t} - \left| u_{ij-\frac{1}{2}k}^{t} \right| \right) u_{ijk}^{t}}{\Delta x_{ij-\frac{1}{2}k}}
\end{array} \right] \Big/ \left(2\Delta x_{ij} \right) \\
+ \Delta t \left[\begin{array}{l}
\dfrac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} \left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} + \left| v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} \right| \right) u_{ijk}^{t} - H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} \left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} + \left| v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} \right| \right) u_{i-1jk}^{t}}{\Delta y_{i-\frac{1}{2}jk}} + \\
\dfrac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} \left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} - \left| v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} \right| \right) u_{i+1jk}^{t} - H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} \left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} - \left| v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} \right| \right) u_{ijk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}
\end{array} \right] \Big/ \left(2\Delta x_{ij} \right) \\
+ \Delta t \left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \left[\begin{array}{l}
\dfrac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}} + \left| w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}} \right| \right) u_{ijk}^{t} - \left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}} + \left| w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}} \right| \right) u_{ijk-1}^{t}}{2H_{ij-\frac{1}{2}k}^{t}} + \\
\dfrac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}} - \left| w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}} \right| \right) u_{ijk+1}^{t} - \left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}} - \left| w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}} \right| \right) u_{ijk}^{t}}{2H_{ij-\frac{1}{2}k}^{t}}
\end{array} \right] \Big/ \left(2\Delta x_{ij} \right)
\end{array} \right] \\
&- \Delta t g \Delta t \sum_{k=kbottom}^{surface} \left(\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \frac{\sum_{k=i}^{k=surflayer} \left(\rho_{ijk} - \rho_{ij-1k} \right) \left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) / 2}{\left(\rho_{ijk} + \rho_{ij-1k} \right) / 2\, \Delta x_{ij-\frac{1}{2}k}} \Big/ \left(2\Delta x_{ij} \right) \right) \\
&+ \Delta t \sum_{k=bottom}^{surface} \left[\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \Delta t \left(\frac{\nu_x}{\Delta x_{ijk}^{2}} \left(u_{ij+1k}^{t} - u_{ijk}^{t} \right) - \frac{\nu_x}{\Delta x_{ij-1k}^{2}} \left(u_{ijk}^{t} - u_{ij-1k}^{t} \right) \right) \Big/ \left(2\Delta x_{ij} \right) \right] \\
&+ \Delta t \sum_{k=bottom}^{surface} \left[\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \Delta t \left(\frac{\nu_y}{\Delta y_{i+\frac{1}{2}jk}^{2}} \left(u_{i+1jk}^{t} - u_{ijk}^{t} \right) - \frac{\nu_y}{\Delta y_{i-\frac{1}{2}jk}^{2}} \left(u_{ijk}^{t} - u_{i-1jk}^{t} \right) \right) \Big/ \left(2\Delta x_{ij} \right) \right] \\
&- \Delta t \Delta t \left(Cf_{ijk} + Cf_{ij-1k} \right) \left| u_{ijk}^{t} \right| u_{ijk}^{t+1} \Big/ \left(2\Delta x_{ij} \right) + \Delta t \frac{\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \Delta t \tau_{ij}^{vento}}{\rho \Delta z_{ijk}} \Big/ \left(2\Delta x_{ij} \right) \\
&+ \Delta t \sum_{k=bottom}^{surface} \frac{f v_{ijk}^{*}}{2\Delta x_{ij}} + \frac{\Delta t}{2\Delta y_{ij}} \left[\sum_{k=bottom}^{surface} \left(v_{ijk}^{t} \left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \right) \right]
\end{aligned}
$$

$$
-\Delta t\sum_{k=bottom}^{surface}\left[u_{ij+1k}^{t}\ \left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)/2\Delta x_{ij}\right]
$$

$$
+\Delta t\sum_{k=bottom}^{surface}\left[\begin{array}{l}
\Delta t\left[\begin{array}{l}\dfrac{H_{ij+1k}^{t}\left(u_{ij+3/2k}^{t}+\left|u_{ij+3/2k}^{t}\right|\right)u_{ij+1k}^{t}-H_{ijk}^{t}\left(u_{ij+1/2k}^{t}+\left|u_{ij+1/2k}^{t}\right|\right)u_{ij+1k}^{t}}{\Delta x_{ij+1/2k}}+\\ \dfrac{H_{ij+1k}^{t}\left(u_{ij+3/2k}^{t}-\left|u_{ij+3/2k}^{t}\right|\right)u_{ij+2k}^{t}-H_{ijk}^{t}\left(u_{ij+1/2k}^{t}-\left|u_{ij+1/2k}^{t}\right|\right)u_{ij+1k}^{t}}{\Delta x_{ij+1/2k}}\end{array}\right]\Big/\left(2\Delta x_{ij}\right)\\
+\Delta t\left[\begin{array}{l}\dfrac{H_{i+1/2j+1/2k}^{t}\left(v_{i+1/2j+1/2k}^{t}+\left|v_{i+1/2j+1/2k}^{t}\right|\right)u_{ij+1k}^{t}-H_{i-1/2j+1/2k}^{t}\left(v_{i-1/2j+1/2k}^{t}+\left|v_{i-1/2j+1/2k}^{t}\right|\right)u_{i-1j+1k}^{t}}{\Delta y_{ij+1/2k}}+\\ \dfrac{H_{i+1/2j+1/2k}^{t}\left(v_{i+1/2j+1/2k}^{t}-\left|v_{i+1/2j+1/2k}^{t}\right|\right)u_{i+1j+1k}^{t}-H_{i-1/2j+1/2k}^{t}\left(v_{i-1/2j+1/2k}^{t}-\left|v_{i-1/2j+1/2k}^{t}\right|\right)u_{ij+1k}^{t}}{\Delta y_{ij+1/2k}}\end{array}\right]\Big/\left(2\Delta x_{ij}\right)\\
+\Delta t\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\left[\begin{array}{l}\dfrac{\left(w_{ij+1/2k+1}^{t-1/2}+\left|w_{ij+1/2k+1}^{t-1/2}\right|\right)u_{ij+1k}^{t}-\left(w_{ij+1/2k}^{t-1/2}+\left|w_{ij+1/2k}^{t-1/2}\right|\right)u_{ij+1k-1}^{t}}{2H_{ij+1/2k}^{t}}+\\ \dfrac{\left(w_{ij+1/2k+1}^{t-1/2}-\left|w_{ij+1/2k+1}^{t-1/2}\right|\right)u_{ij+1k+1}^{t}-\left(w_{ij+1/2k}^{t-1/2}-\left|w_{ij+1/2k}^{t-1/2}\right|\right)u_{ij+1k}^{t}}{2H_{ij+1/2k}^{t}}\end{array}\right]\Big/\left(2\Delta x_{ij}\right)
\end{array}\right]
$$

$$
+\Delta tg\Delta t\sum_{k=bottom}^{surface}\left(\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\frac{\sum\limits_{k=i}^{k=surflayer}\left(\rho_{ij+1k}-\rho_{ijk}\right)\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)/2}{\left(\rho_{ij+1k}+\rho_{ijk}\right)/2\,\Delta x_{ij+1/2k}}\Big/\left(2\Delta x_{ij}\right)\right)
$$

$$
-\Delta t\sum_{k=bottom}^{surface}\left[\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\Delta t\left(\frac{\nu_{x}}{\Delta x_{ij+1k}^{2}}\left(u_{ij+2k}^{t}-u_{ij+1k}^{t}\right)-\frac{\nu_{x}}{\Delta x_{ijk}^{2}}\left(u_{ij+1k}^{t}-u_{ijk}^{t}\right)\right)\Big/\left(2\Delta x_{ij}\right)\right]
$$

$$
-\Delta t\sum_{k=bottom}^{surface}\left[\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\Delta t\left(\frac{\nu_{y}}{\Delta y_{i+1/2j+1k}^{2}}\left(u_{i+1j+1k}^{t}-u_{ij+1k}^{t}\right)-\frac{\nu_{y}}{\Delta y_{i-1/2j+1k}^{2}}\left(u_{ij+1k}^{t}-u_{i-1j+1k}^{t}\right)\right)\Big/\left(2\Delta x_{ij}\right)\right]
$$

$$
+\Delta t\Delta t\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u_{ij+1k}^{t}\right|u_{ij+1k}^{t+1}/\left(4\Delta x_{ij}\right)-\Delta t\frac{\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\Delta t\,\tau_{j}^{vento}}{\rho\Delta z_{ij+1k}}\Big/\left(2\Delta x_{ij}\right)
$$

$$
+\Delta t\sum_{k=bottom}^{surface}\left(\frac{FlowInput_{ijk}-FlowOutput_{ijk}}{\Delta x_{ijk}\Delta y_{ijk}}\right)
$$

$$
-\Delta t\sum_{k=bottom}^{surface}\frac{f\,v_{ij+1k}^{*}}{2\Delta x_{ij}}-\frac{\Delta t}{2\Delta y_{ij+1k}}\left[\sum_{k=bottom}^{surface}\left(v_{ij+1k}^{t}\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\right)\right]
$$

(30)

Equation 31 is obtained from 30, placing all implicit terms on the left side of 30 and explicit terms on its right side.

$$
\begin{aligned}
&\xi_{ij}^{t+\frac{1}{2}} + \Delta t g \Delta t \sum_{k=bottom}^{surface} \left(\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \frac{\xi_{ij}^{t+\frac{1}{2}} - \xi_{ij-1}^{t+\frac{1}{2}}}{\Delta x_{ij-\frac{1}{2}}} \Big/ \left(2\Delta x_{ij} \right) \right) \\
&-\Delta t g \Delta t \sum_{k=bottom}^{surface} \left(\left(H_{ij+1k}^{t} + H_{ijk}^{t} \right) \frac{\xi_{ij+1}^{t+\frac{1}{2}} - \xi_{ij}^{t+\frac{1}{2}}}{\Delta x_{ij+\frac{1}{2}}} \Big/ \left(2\Delta x_{ij} \right) \right) \\
&+\Delta t \Delta t \left(Cf_{ijk} + Cf_{ij-1k} \right) \left| u_{ijk}^{t} \right| u_{ijk}^{t+1} \Big/ \left(2\Delta x_{ij} \right) \\
&-\Delta t \Delta t \left(Cf_{ij+1k} + Cf_{ijk} \right) \left| u_{ij+1k}^{t} \right| u_{ij+1k}^{t+1} \Big/ \left(2\Delta x_{ij} \right) \\
&= \xi_{ij}^{t} + \Delta t \sum_{k=bottom}^{surface} \left[u_{ijk}^{t} \left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \Big/ 2\Delta x_{ij} \right] \\
&-\Delta t \sum_{k=bottom}^{surface} \left[
\begin{gathered}
\Delta t \left[\begin{gathered} \frac{ H_{ijk}^{t} \left(u_{ij+\frac{1}{2}k}^{t} + \left| u_{ij+\frac{1}{2}k}^{t} \right| \right) u_{ijk}^{t} - H_{ij-1k}^{t} \left(u_{ij-\frac{1}{2}k}^{t} + \left| u_{ij-\frac{1}{2}k}^{t} \right| \right) u_{ij-1k}^{t} }{ \Delta x_{ij-\frac{1}{2}k} } + \\ \frac{ H_{ijk}^{t} \left(u_{ij+\frac{1}{2}k}^{t} - \left| u_{ij+\frac{1}{2}k}^{t} \right| \right) u_{ij+1k}^{t} - H_{ij-1k}^{t} \left(u_{ij-\frac{1}{2}k}^{t} - \left| u_{ij-\frac{1}{2}k}^{t} \right| \right) u_{ijk}^{t} }{ \Delta x_{ij-\frac{1}{2}k} } \end{gathered} \right] \Big/ \left(2\Delta x_{ij} \right) \\
+\Delta t \left[\begin{gathered} \frac{ H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} \left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} + \left| v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} \right| \right) u_{ijk}^{t} - H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} \left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} + \left| v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} \right| \right) u_{i-1jk}^{t} }{ \Delta y_{i-\frac{1}{2}jk} } + \\ \frac{ H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} \left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} - \left| v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} \right| \right) u_{i+1jk}^{t} - H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} \left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} - \left| v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} \right| \right) u_{ijk}^{t} }{ \Delta y_{i-\frac{1}{2}jk} } \end{gathered} \right] \Big/ \left(2\Delta x_{ij} \right) \\
+\Delta t \left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \left[\begin{gathered} \frac{ \left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}} + \left| w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}} \right| \right) u_{ijk}^{t} - \left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}} + \left| w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}} \right| \right) u_{ijk-1}^{t} }{ 2H_{ij-\frac{1}{2}k}^{t} } + \\ \frac{ \left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}} - \left| w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}} \right| \right) u_{ijk+1}^{t} - \left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}} - \left| w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}} \right| \right) u_{ijk}^{t} }{ 2H_{ij-\frac{1}{2}k}^{t} } \end{gathered} \right] \Big/ \left(2\Delta x_{ij} \right)
\end{gathered}
\right] \\
&-\Delta t g \Delta t \sum_{k=kbottom}^{surface} \left(\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \frac{ \sum_{k=i}^{k=surflayer} \left(\rho_{ijk} - \rho_{ij-1k} \right) \left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \big/ 2 }{ \left(\rho_{ijk} + \rho_{ij-1k} \right) \big/ 2\, \Delta x_{ij-\frac{1}{2}k} } \Big/ \left(2\Delta x_{ij} \right) \right) \\
&+\Delta t \sum_{k=bottom}^{surface} \left[\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \Delta t \left(\frac{\nu_x}{\Delta x_{ijk}^{2}} \left(u_{ij+1k}^{t} - u_{ijk}^{t} \right) - \frac{\nu_x}{\Delta x_{ij-1k}^{2}} \left(u_{ijk}^{t} - u_{ij-1k}^{t} \right) \right) \Big/ \left(2\Delta x_{ij} \right) \right] \\
&+\Delta t \sum_{k=bottom}^{surface} \left[\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \Delta t \left(\frac{\nu_y}{\Delta y_{ij+\frac{1}{2}k}^{2}} \left(u_{i+1jk}^{t} - u_{ijk}^{t} \right) - \frac{\nu_y}{\Delta y_{ij-\frac{1}{2}k}^{2}} \left(u_{ijk}^{t} - u_{i-1jk}^{t} \right) \right) \Big/ \left(2\Delta x_{ij} \right) \right] \\
&+\Delta t \frac{ \left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \Delta t \, \tau_{ij}^{vento} }{ \rho \Delta z_{ijk} } \Big/ \left(2\Delta x_{ij} \right) \\
&+\Delta t \sum_{k=bottom}^{surface} \frac{ f v_{ijk}^{*} }{ 2\Delta x_{ij} } + \frac{\Delta t}{2\Delta y_{ij}} \left[\sum_{k=bottom}^{surface} \left(v_{ijk}^{t} \left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \right) \right]
\end{aligned}
$$

$$-\Delta t\sum_{k=bottom}^{surface}\left[u_{ij+1k}^{t}\ \left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)/2\Delta x_{ij}\right]$$

$$+\Delta t\sum_{k=bottom}^{surface}\left[\begin{array}{l}\Delta t\left[\begin{array}{l}\dfrac{H_{ij+1k}^{t}\left(u_{ij+3/2k}^{t}+\left|u_{ij+3/2k}^{t}\right|\right)u_{ij+1k}^{t}-H_{ijk}^{t}\left(u_{ij+1/2k}^{t}+\left|u_{ij+1/2k}^{t}\right|\right)u_{ij+1k}^{t}}{\Delta x_{ij+1/2k}}+\\ \dfrac{H_{ij+1k}^{t}\left(u_{ij+3/2k}^{t}-\left|u_{ij+3/2k}^{t}\right|\right)u_{ij+2k}^{t}-H_{ijk}^{t}\left(u_{ij+1/2k}^{t}-\left|u_{ij+1/2k}^{t}\right|\right)u_{ij+1k}^{t}}{\Delta x_{ij+1/2k}}\end{array}\right]\Big/\left(2\Delta x_{ij}\right)\\ +\Delta t\left[\begin{array}{l}\dfrac{H_{i+1/2j+1/2k}^{t}\left(v_{i+1/2j+1/2k}^{t}+\left|v_{i+1/2j+1/2k}^{t}\right|\right)u_{ij+1k}^{t}-H_{i-1/2j+1/2k}^{t}\left(v_{i-1/2j+1/2k}^{t}+\left|v_{i-1/2j+1/2k}^{t}\right|\right)u_{i-1j+1k}^{t}}{\Delta y_{ij+1/2k}}+\\ \dfrac{H_{i+1/2j+1/2k}^{t}\left(v_{i+1/2j+1/2k}^{t}-\left|v_{i+1/2j+1/2k}^{t}\right|\right)u_{i+1j+1k}^{t}-H_{i-1/2j+1/2k}^{t}\left(v_{i-1/2j+1/2k}^{t}-\left|v_{i-1/2j+1/2k}^{t}\right|\right)u_{ij+1k}^{t}}{\Delta y_{ij+1/2k}}\end{array}\right]\Big/\left(2\Delta x_{ij}\right)\\ +\Delta t\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\left[\begin{array}{l}\dfrac{\left(w_{ij+1/2k+1}^{t-1/2}+\left|w_{ij+1/2k+1}^{t-1/2}\right|\right)u_{ij+1k}^{t}-\left(w_{ij+1/2k}^{t-1/2}+\left|w_{ij+1/2k}^{t-1/2}\right|\right)u_{ij+1k-1}^{t}}{2H_{ij+1/2k}^{t}}+\\ \dfrac{\left(w_{ij+1/2k+1}^{t-1/2}-\left|w_{ij+1/2k+1}^{t-1/2}\right|\right)u_{ij+1k+1}^{t}-\left(w_{ij+1/2k}^{t-1/2}-\left|w_{ij+1/2k}^{t-1/2}\right|\right)u_{ij+1k}^{t}}{2H_{ij+1/2k}^{t}}\end{array}\right]\Big/\left(2\Delta x_{ij}\right)\end{array}\right]$$

$$+\Delta tg\Delta t\sum_{k=bottom}^{surface}\left(\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\frac{\sum\limits_{k=i}^{k=surflayer}\left(\rho_{ij+1k}-\rho_{ijk}\right)\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)/2}{\left(\rho_{ij+1k}+\rho_{ijk}\right)/2\,\Delta x_{ij+1/2k}}\Big/\left(2\Delta x_{ij}\right)\right)$$

$$-\Delta t\sum_{k=bottom}^{surface}\left[\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\Delta t\left(\frac{\nu_x}{\Delta x_{ij+1k}^{2}}\left(u_{ij+2k}^{t}-u_{ij+1k}^{t}\right)-\frac{\nu_x}{\Delta x_{ijk}^{2}}\left(u_{ij+1k}^{t}-u_{ijk}^{t}\right)\right)\Big/\left(2\Delta x_{ij}\right)\right]$$

$$-\Delta t\sum_{k=bottom}^{surface}\left[\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\Delta t\left(\frac{\nu_y}{\Delta y_{i+1/2j+1k}^{2}}\left(u_{i+1j+1k}^{t}-u_{ij+1k}^{t}\right)-\frac{\nu_y}{\Delta y_{i-1/2j+1k}^{2}}\left(u_{ij+1k}^{t}-u_{i-1j+1k}^{t}\right)\right)\Big/\left(2\Delta x_{ij}\right)\right]$$

$$-\Delta t\frac{\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\Delta t\,\tau_{j}^{vento}}{\rho\Delta z_{ij+1k}}\Big/\left(2\Delta x_{ij}\right)$$

$$+\Delta t\sum_{k=bottom}^{surface}\left(\frac{FlowInput_{ijk}-FlowOutput_{ijk}}{\Delta x_{ijk}\Delta y_{ijk}}\right)$$

$$-\Delta t\sum_{k=bottom}^{surface}\frac{f\,v_{ij+1k}^{*}}{2\Delta x_{ij}}-\frac{\Delta t}{2\Delta y_{ij+1k}}\left[\sum_{k=bottom}^{surface}\left(v_{ij+1k}^{t}\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\right)\right]$$

(31)

Equation 32 is obtained by inserting equation 29 into equation 31.

$$\xi_{ij}^{t+1/2}+\Delta tg\Delta t\sum_{k=bottom}^{surface}\left(\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\frac{\xi_{ij}^{t+1/2}-\xi_{ij-1}^{t+1/2}}{\Delta x_{ij-1/2}}\Big/\left(2\Delta x_{ij}\right)\right)$$

$$-\Delta tg\Delta t\sum_{k=bottom}^{surface}\left(\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\frac{\xi_{ij+1}^{t+1/2}-\xi_{ij}^{t+1/2}}{\Delta x_{ij+1/2}}\Big/\left(2\Delta x_{ij}\right)\right)$$

$$+\Delta t\Delta t\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u_{ijk}^{t}\right|\frac{\left[-g\Delta t\left(\dfrac{\xi_{ij}^{t+1/2}-\xi_{ij-1}^{t+1/2}}{\Delta x_{ij-1/2k}}\right)+Other\_terms\right]}{\left(1+\dfrac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u_{ijk}^{t}\right|\right)}\Big/\left(2\Delta x_{ij}\right)$$

$$-\Delta t\Delta t\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u_{ij+1k}^{t}\right|\frac{\left[-g\Delta t\left(\dfrac{\xi_{ij+1}^{t+1/2}-\xi_{ij}^{t+1/2}}{\Delta x_{ij+1/2k}}\right)+Other\_terms\right]}{\left(1+\dfrac{\Delta t}{\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)}\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u_{ij+1k}^{t}\right|\right)}\Big/\left(2\Delta x_{ij}\right)$$

$$=see\_above$$

(32)

The next step is to simplify the left side of 32 and moving back to its right side all terms that do not have terms in t+1, obtaining 33.

$$
\begin{aligned}
&-\frac{g\Delta t^2}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H^t_{ijk}+H^t_{ij-1k}\right)\frac{\xi^{t+1/2}_{ij-1}}{\Delta x_{ij-1/2}}\right)\\
&+g\Delta t^3\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u^t_{ijk}\right|\frac{\left(\dfrac{\xi^{t+1/2}_{ij-1}}{\Delta x_{ij-1/2k}}\right)}{\left(1+\dfrac{\Delta t}{\left(H^t_{ijk}+H^t_{ij-1k}\right)}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u^t_{ijk}\right|\right)}\Bigg/2\Delta x_{ij}\\
&+\xi^{t+1/2}_{ij}+\frac{g\Delta t^2}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H^t_{ijk}+H^t_{ij-1k}\right)\frac{\xi^{t+1/2}_{ij}}{\Delta x_{ij-1/2}}\right)+\frac{g\Delta t^2}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H^t_{ij+1k}+H^t_{ijk}\right)\frac{\xi^{t+1/2}_{ij}}{\Delta x_{ij+1/2}}\right)\\
&-g\Delta t^3\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u^t_{ijk}\right|\frac{\left(\dfrac{\xi^{t+1/2}_{ij}}{\Delta x_{ij-1/2k}}\right)}{\left(1+\dfrac{\Delta t}{\left(H^t_{ijk}+H^t_{ij-1k}\right)}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u^t_{ijk}\right|\right)}\Bigg/\left(2\Delta x_{ij}\right)\\
&-g\Delta t^3\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u^t_{ij+1k}\right|\frac{\left(\dfrac{\xi^{t+1/2}_{ij}}{\Delta x_{ij+1/2k}}\right)}{\left(1+\dfrac{\Delta t}{\left(H^t_{ij+1k}+H^t_{ijk}\right)}\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u^t_{ij+1k}\right|\right)}\Bigg/\left(2\Delta x_{ij}\right)\\
&-\frac{g\Delta t^2}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H^t_{ij+1k}+H^t_{ijk}\right)\frac{\xi^{t+1/2}_{ij+1}}{\Delta x_{ij+1/2}}\right)\\
&+g\Delta t^3\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u^t_{ij+1k}\right|\frac{\left(\dfrac{\xi^{t+1/2}_{ij+1}}{\Delta x_{ij+1/2k}}\right)}{\left(1+\dfrac{\Delta t}{\left(H^t_{ij+1k}+H^t_{ijk}\right)}\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u^t_{ij+1k}\right|\right)}\Bigg/\left(2\Delta x_{ij}\right)\\
&=see\_above\\
&-\Delta t^2\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u^t_{ijk}\right|\frac{\left[Other\_terms\right]}{\left(1+\dfrac{\Delta t}{\left(H^t_{ijk}+H^t_{ij-1k}\right)}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u^t_{ijk}\right|\right)}\Bigg/\left(2\Delta x_{ij}\right)\\
&+\Delta t^2\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u^t_{ij+1k}\right|\frac{\left[Other\_terms\right]}{\left(1+\dfrac{\Delta t}{\left(H^t_{ij+1k}+H^t_{ijk}\right)}\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u^t_{ij+1k}\right|\right)}\Bigg/\left(2\Delta x_{ij}\right)
\end{aligned}
$$

(33)

Equation 34 is the complete version of 33 and it corresponds to the continuity equation for “normal” cell columns. Equations 35 and 36 are modified versions of 34 for upstream and the downstream boundaries, respectively.

$$-\frac{g\Delta t^{2}}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H^{t}_{ijk}+H^{t}_{ij-1k}\right)\frac{\xi^{t+1/2}_{ij-1}}{\Delta x_{ij-1/2}}\right)$$

$$+g\Delta t^{3}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u^{t}_{ijk}\right|\frac{\left(\dfrac{\xi^{t+1/2}_{ij-1}}{\Delta x_{ij-1/2k}}\right)}{\left(1+\dfrac{\Delta t}{\left(H^{t}_{ijk}+H^{t}_{ij-1k}\right)}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u^{t}_{ijk}\right|\right)}\Bigg/2\Delta x_{ij}$$

$$+\xi^{t+1/2}_{ij}+\frac{g\Delta t^{2}}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H^{t}_{ijk}+H^{t}_{ij-1k}\right)\frac{\xi^{t+1/2}_{ij}}{\Delta x_{ij-1/2}}\right)+\frac{g\Delta t^{2}}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H^{t}_{ij+1k}+H^{t}_{ijk}\right)\frac{\xi^{t+1/2}_{ij}}{\Delta x_{ij+1/2}}\right)$$

$$-g\Delta t^{3}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u^{t}_{ijk}\right|\frac{\left(\dfrac{\xi^{t+1/2}_{ij}}{\Delta x_{ij-1/2k}}\right)}{\left(1+\dfrac{\Delta t}{\left(H^{t}_{ijk}+H^{t}_{ij-1k}\right)}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u^{t}_{ijk}\right|\right)}\Bigg/\left(2\Delta x_{ij}\right)$$

$$-g\Delta t^{3}\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u^{t}_{ij+1k}\right|\frac{\left(\dfrac{\xi^{t+1/2}_{ij}}{\Delta x_{ij+1/2k}}\right)}{\left(1+\dfrac{\Delta t}{\left(H^{t}_{ij+1k}+H^{t}_{ijk}\right)}\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u^{t}_{ij+1k}\right|\right)}\Bigg/\left(2\Delta x_{ij}\right)$$

$$-\frac{g\Delta t^{2}}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H^{t}_{ij+1k}+H^{t}_{ijk}\right)\frac{\xi^{t+1/2}_{ij+1}}{\Delta x_{ij+1/2}}\right)$$

$$+g\Delta t^{3}\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u^{t}_{ij+1k}\right|\frac{\left(\dfrac{\xi^{t+1/2}_{ij+1}}{\Delta x_{ij+1/2k}}\right)}{\left(1+\dfrac{\Delta t}{\left(H^{t}_{ij+1k}+H^{t}_{ijk}\right)}\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u^{t}_{ij+1k}\right|\right)}\Bigg/\left(2\Delta x_{ij}\right)=$$

$$
\begin{aligned}
&= \xi_{ij}^{t} + \Delta t \sum_{k=bottom}^{surface} \left[u_{ijk}^{t} \; \left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) / 2\Delta x_{ij} \right] \\
&- \Delta t \sum_{k=bottom}^{surface} \left[\begin{array}{l}
\Delta t \left[\begin{array}{l} \dfrac{H_{ijk}^{t}\left(u_{ij+\frac{1}{2}k}^{t} + \left|u_{ij+\frac{1}{2}k}^{t}\right|\right) u_{ijk}^{t} - H_{ij-1k}^{t}\left(u_{ij-\frac{1}{2}k}^{t} + \left|u_{ij-\frac{1}{2}k}^{t}\right|\right) u_{ij-1k}^{t}}{\Delta x_{ij-\frac{1}{2}k}} + \\ \dfrac{H_{ijk}^{t}\left(u_{ij+\frac{1}{2}k}^{t} - \left|u_{ij+\frac{1}{2}k}^{t}\right|\right) u_{ij+1k}^{t} - H_{ij-1k}^{t}\left(u_{ij-\frac{1}{2}k}^{t} - \left|u_{ij-\frac{1}{2}k}^{t}\right|\right) u_{ijk}^{t}}{\Delta x_{ij-\frac{1}{2}k}} \end{array} \right] \Big/ \left(2\Delta x_{ij}\right) \\
+ \Delta t \left[\begin{array}{l} \dfrac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} + \left|v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right) u_{ijk}^{t} - H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} + \left|v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right) u_{i-1jk}^{t}}{\Delta y_{i-\frac{1}{2}jk}} + \\ \dfrac{H_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t} - \left|v_{i+\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right) u_{i+1jk}^{t} - H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} - \left|v_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right) u_{ijk}^{t}}{\Delta y_{i-\frac{1}{2}jk}} \end{array} \right] \Big/ \left(2\Delta x_{ij}\right) \\
+ \Delta t \left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \left[\begin{array}{l} \dfrac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}} + \left|w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}\right|\right) u_{ijk}^{t} - \left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}} + \left|w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}\right|\right) u_{ijk-1}^{t}}{2H_{ij-\frac{1}{2}k}^{t}} + \\ \dfrac{\left(w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}} - \left|w_{ij-\frac{1}{2}k+1}^{t-\frac{1}{2}}\right|\right) u_{ijk+1}^{t} - \left(w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}} - \left|w_{ij-\frac{1}{2}k}^{t-\frac{1}{2}}\right|\right) u_{ijk}^{t}}{2H_{ij-\frac{1}{2}k}^{t}} \end{array} \right] \Big/ \left(2\Delta x_{ij}\right)
\end{array} \right] \\
&- \Delta t g \Delta t \sum_{k=kbottom}^{surface} \left(\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \frac{\sum_{k=i}^{k=surflayer} \left(\rho_{ijk} - \rho_{ij-1k}\right)\left(H_{ijk}^{t} + H_{ij-1k}^{t}\right)/2}{\left(\rho_{ijk} + \rho_{ij-1k}\right)/2\, \Delta x_{ij-\frac{1}{2}k}} \Big/ \left(2\Delta x_{ij}\right) \right) \\
&+ \Delta t \sum_{k=bottom}^{surface} \left[\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \Delta t \left(\frac{\nu_x}{\Delta x_{ijk}^{2}} \left(u_{ij+1k}^{t} - u_{ijk}^{t} \right) - \frac{\nu_x}{\Delta x_{ij-1k}^{2}} \left(u_{ijk}^{t} - u_{ij-1k}^{t} \right) \right) \Big/ \left(2\Delta x_{ij}\right) \right] \\
&+ \Delta t \sum_{k=bottom}^{surface} \left[\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \Delta t \left(\frac{\nu_y}{\Delta y_{i+\frac{1}{2}jk}^{2}} \left(u_{i+1jk}^{t} - u_{ijk}^{t} \right) - \frac{\nu_y}{\Delta y_{i-\frac{1}{2}jk}^{2}} \left(u_{ijk}^{t} - u_{i-1jk}^{t} \right) \right) \Big/ \left(2\Delta x_{ij}\right) \right] \\
&+ \Delta t \frac{\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \Delta t \tau_{ij}^{vento}}{\rho \Delta z_{ijk}} \Big/ \left(2\Delta x_{ij}\right) \\
&+ \Delta t \sum_{k=bottom}^{surface} \frac{f v_{ijk}^{*}}{4\Delta x_{ij}} + \frac{\Delta t}{2\Delta y_{ij}} \left[\sum_{k=bottom}^{surface} \left(v_{ijk}^{t} \left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \right) \right] \\
&- \Delta t^{2} \left(Cf_{ijk} + Cf_{ij-1k} \right) \left| u_{ijk}^{t} \right| \frac{\left[Other\_terms \right]}{\left(1 + \dfrac{\Delta t}{\left(H_{ijk}^{t} + H_{ij-1k}^{t} \right)} \left(Cf_{ijk} + Cf_{ij-1k} \right) \left| u_{ijk}^{t} \right| \right)} \Big/ \left(2\Delta x_{ij}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\Delta t\sum_{k=bottom}^{surface}\left[u^{t}_{ij+1k}\;\left(H^{t}_{ij+1k}+H^{t}_{ijk}\right)\Big/2\Delta x_{ij}\right]\\
&\left[\begin{aligned}
&\Delta t\left[\begin{aligned}&\frac{H^{t}_{ij+1k}\left(u^{t}_{ij+3/2k}+\left|u^{t}_{ij+3/2k}\right|\right)u^{t}_{ij+1k}-H^{t}_{ijk}\left(u^{t}_{ij+1/2k}+\left|u^{t}_{ij+1/2k}\right|\right)u^{t}_{ij+1k}}{\Delta x_{ij+1/2k}}+\\&\frac{H^{t}_{ij+1k}\left(u^{t}_{ij+3/2k}-\left|u^{t}_{ij+3/2k}\right|\right)u^{t}_{ij+2k}-H^{t}_{ijk}\left(u^{t}_{ij+1/2k}-\left|u^{t}_{ij+1/2k}\right|\right)u^{t}_{ij+1k}}{\Delta x_{ij+1/2k}}\end{aligned}\right]\Bigg/\left(2\Delta x_{ij}\right)\\
+\Delta t\sum_{k=bottom}^{surface}\;&+\Delta t\left[\begin{aligned}&\frac{H^{t}_{i+1/2j+1/2k}\left(v^{t}_{i+1/2j+1/2k}+\left|v^{t}_{i+1/2j+1/2k}\right|\right)u^{t}_{ij+1k}-H^{t}_{i-1/2j+1/2k}\left(v^{t}_{i-1/2j+1/2k}+\left|v^{t}_{i-1/2j+1/2k}\right|\right)u^{t}_{i-1j+1k}}{\Delta y_{ij+1/2k}}+\\&\frac{H^{t}_{i+1/2j+1/2k}\left(v^{t}_{i+1/2j+1/2k}-\left|v^{t}_{i+1/2j+1/2k}\right|\right)u^{t}_{i+1j+1k}-H^{t}_{i-1/2j+1/2k}\left(v^{t}_{i-1/2j+1/2k}-\left|v^{t}_{i-1/2j+1/2k}\right|\right)u^{t}_{ij+1k}}{\Delta y_{ij+1/2k}}\end{aligned}\right]\Bigg/\left(2\Delta x_{ij}\right)\\
&+\Delta t\left(H^{t}_{ij+1k}+H^{t}_{ijk}\right)\left[\begin{aligned}&\frac{\left(w^{t-1/2}_{ij+1/2k+1}+\left|w^{t-1/2}_{ij+1/2k+1}\right|\right)u^{t}_{ij+1k}-\left(w^{t-1/2}_{ij+1/2k}+\left|w^{t-1/2}_{ij+1/2k}\right|\right)u^{t}_{ij+1k-1}}{2H^{t}_{ij+1/2k}}+\\&\frac{\left(w^{t-1/2}_{ij+1/2k+1}-\left|w^{t-1/2}_{ij+1/2k+1}\right|\right)u^{t}_{ij+1k+1}-\left(w^{t-1/2}_{ij+1/2k}-\left|w^{t-1/2}_{ij+1/2k}\right|\right)u^{t}_{ij+1k}}{2H^{t}_{ij+1/2k}}\end{aligned}\right]\Bigg/\left(2\Delta x_{ij}\right)
\end{aligned}\right]\\
&+\Delta tg\Delta t\sum_{k=bottom}^{surface}\left(\left(H^{t}_{ij+1k}+H^{t}_{ijk}\right)\frac{\sum\limits_{k=i}^{k=surflayer}\left(\rho_{ij+1k}-\rho_{ijk}\right)\left(H^{t}_{ij+1k}+H^{t}_{ijk}\right)\big/2}{\left(\rho_{ij+1k}+\rho_{ijk}\right)\big/2\,\Delta x_{ij+1/2k}}\Bigg/\left(2\Delta x_{ij}\right)\right)\\
&-\Delta t\sum_{k=bottom}^{surface}\left[\left(H^{t}_{ij+1k}+H^{t}_{ijk}\right)\Delta t\left(\frac{\nu_x}{\Delta x^2_{ij+1k}}\left(u^{t}_{ij+2k}-u^{t}_{ij+1k}\right)-\frac{\nu_x}{\Delta x^2_{ijk}}\left(u^{t}_{ij+1k}-u^{t}_{ijk}\right)\right)\Big/\left(2\Delta x_{ij}\right)\right]\\
&-\Delta t\sum_{k=bottom}^{surface}\left[\left(H^{t}_{ij+1k}+H^{t}_{ijk}\right)\Delta t\left(\frac{\nu_y}{\Delta y^2_{i+1/2j+1k}}\left(u^{t}_{i+1j+1k}-u^{t}_{ij+1k}\right)-\frac{\nu_y}{\Delta y^2_{i-1/2j+1k}}\left(u^{t}_{ij+1k}-u^{t}_{i-1j+1k}\right)\right)\Big/\left(2\Delta x_{ij}\right)\right]\\
&-\Delta t\,\frac{\left(H^{t}_{ij+1k}+H^{t}_{ijk}\right)\Delta t\,\tau^{vento}_{j}}{\rho\Delta z_{ij+1k}}\Bigg/\left(2\Delta x_{ij}\right)\\
&+\Delta t\sum_{k=bottom}^{surface}\left(\frac{FlowInput_{ijk}-FlowOutput_{ijk}}{\Delta x_{ijk}\Delta y_{ijk}}\right)\\
&-\Delta t\sum_{k=bottom}^{surface}\frac{f\,v^{*}_{ij+1k}}{2\Delta x_{ij}}-\frac{\Delta t}{2\Delta y_{ij+1k}}\left[\sum_{k=bottom}^{surface}\left(v^{t}_{ij+1k}\left(H^{t}_{ij+1k}+H^{t}_{ijk}\right)\right)\right]\\
&+\Delta t^{2}\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u^{t}_{ij+1k}\right|\frac{\left[Other\_terms\right]}{\left(1+\frac{\Delta t}{\left(H^{t}_{ij+1k}+H^{t}_{ijk}\right)}\left(Cf_{ij+1k}+Cf_{ijk}\right)\left|u^{t}_{ij+1k}\right|\right)}\Bigg/\left(2\Delta x_{ij}\right)
\end{aligned}
$$

(34)

At the upstream boundary:

$$+\xi_{ij}^{t+1/2}+\frac{g\Delta t^{2}}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\frac{\xi_{ij}^{t+1/2}}{\Delta x_{ij+1/2}}\right)$$

$$-g\Delta t^{3}\left(Cf_{\ ij+1k}+Cf_{\ ijk}\right)\left|u_{ij+1k}^{t}\right|\frac{\left(\dfrac{\xi_{ij}^{t+1/2}}{\Delta x_{ij+1/2k}}\right)}{\left(1+\dfrac{\Delta t}{\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)}\left(Cf_{\ ij+1k}+Cf_{\ ijk}\right)\left|u_{ij+1k}^{t}\right|\right)}\Bigg/\left(2\Delta x_{ij}\right)$$

$$-\frac{g\Delta t^{2}}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\frac{\xi_{ij+1}^{t+1/2}}{\Delta x_{ij+1/2}}\right)$$

$$+g\Delta t^{3}\left(Cf_{\ ij+1k}+Cf_{\ ijk}\right)\left|u_{ij+1k}^{t}\right|\frac{\left(\dfrac{\xi_{ij+1}^{t+1/2}}{\Delta x_{ij+1/2k}}\right)}{\left(1+\dfrac{\Delta t}{\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)}\left(Cf_{\ ij+1k}+Cf_{\ ijk}\right)\left|u_{ij+1k}^{t}\right|\right)}\Bigg/\left(2\Delta x_{ij}\right)=$$

$$
\xi_{ij}^{t} + \Delta t \sum_{k=bottom}^{surface} \left(\frac{u_{iriverk}^{t+1} H_{ij}^{t} \Delta y_{iriverk}}{2\Delta x_{ijk} \Delta y_{ij+\frac{1}{2}k}} \right)
$$

$$
+\Delta t \sum_{k=bottom}^{surface} \left(\frac{u_{iriverk}^{t} H_{ij}^{t} \Delta y_{iriverk}}{2\Delta x_{ijk} \Delta y_{ij+\frac{1}{2}k}} \right) - \Delta t \sum_{k=bottom}^{surface} \left[u_{ij+1k}^{t} \left(H_{ij+1k}^{t} + H_{ijk}^{t} \right) \Big/ \left(2\Delta x_{ij} \right) \right]
$$

$$
+\Delta t \sum_{k=bottom}^{surface} \left[\begin{array}{l}
\Delta t \left[\begin{array}{l} \dfrac{H_{ij+1k}^{t} \left(u_{ij+\frac{3}{2}k}^{t} + \left| u_{ij+\frac{3}{2}k}^{t} \right| \right) u_{ij+1k}^{t} - H_{ijk}^{t} \left(u_{ij+\frac{1}{2}k}^{t} + \left| u_{ij+\frac{1}{2}k}^{t} \right| \right) u_{ijk}^{t}}{\Delta x_{ij+\frac{1}{2}k}} + \\ \dfrac{H_{ij+1k}^{t} \left(u_{ij+\frac{3}{2}k}^{t} - \left| u_{ij+\frac{3}{2}k}^{t} \right| \right) u_{ij+2k}^{t} - H_{ijk}^{t} \left(u_{ij+\frac{1}{2}k}^{t} - \left| u_{ij+\frac{1}{2}k}^{t} \right| \right) u_{ij+1k}^{t}}{\Delta x_{ij+\frac{1}{2}k}} \end{array} \right] \Bigg/ \left(2\Delta x_{ij} \right) \\
+\Delta t \left[\begin{array}{l} \dfrac{H_{i+\frac{1}{2}j+\frac{1}{2}k}^{t} \left(v_{i+\frac{1}{2}j+\frac{1}{2}k}^{t} + \left| v_{i+\frac{1}{2}j+\frac{1}{2}k}^{t} \right| \right) u_{ij+1k}^{t} - H_{i-\frac{1}{2}j+\frac{1}{2}k}^{t} \left(v_{i-\frac{1}{2}j+\frac{1}{2}k}^{t} + \left| v_{i-\frac{1}{2}j+\frac{1}{2}k}^{t} \right| \right) u_{i-1j+1k}^{t}}{\Delta y_{ij+\frac{1}{2}k}} + \\ \dfrac{H_{i+\frac{1}{2}j+\frac{1}{2}1k}^{t} \left(v_{i+\frac{1}{2}j+\frac{1}{2}k}^{t} - \left| v_{i+\frac{1}{2}j+\frac{1}{2}k}^{t} \right| \right) u_{i+1j+1k}^{t} - H_{i-\frac{1}{2}j+\frac{1}{2}k}^{t} \left(v_{i-\frac{1}{2}j+\frac{1}{2}k}^{t} - \left| v_{i-\frac{1}{2}j+\frac{1}{2}k}^{t} \right| \right) u_{ij+1k}^{t}}{\Delta y_{ij+\frac{1}{2}k}} \end{array} \right] \Bigg/ \left(2\Delta x_{ij} \right) \\
+\Delta t \left(H_{ij+1k}^{t} + H_{ijk}^{t} \right) \left[\begin{array}{l} \dfrac{\left(w_{ij+\frac{1}{2}k+1}^{t-\frac{1}{2}} + \left| w_{ij+\frac{1}{2}k+1}^{t-\frac{1}{2}} \right| \right) u_{ij+1k}^{t} - \left(w_{ij+\frac{1}{2}k}^{t-\frac{1}{2}} + \left| w_{ij+\frac{1}{2}k}^{t-\frac{1}{2}} \right| \right) u_{ij+1k-1}^{t}}{2H_{ij+\frac{1}{2}1k}^{t}} + \\ \dfrac{\left(w_{ij+\frac{1}{2}k+1}^{t-\frac{1}{2}} - \left| w_{ij+\frac{1}{2}k+1}^{t-\frac{1}{2}} \right| \right) u_{ij+1k+1}^{t} - \left(w_{ij+\frac{1}{2}k}^{t-\frac{1}{2}} - \left| w_{ij+\frac{1}{2}k}^{t-\frac{1}{2}} \right| \right) u_{ij+1k}^{t}}{2H_{ij+\frac{1}{2}1k}^{t}} \end{array} \right] \Bigg/ \left(2\Delta x_{ij} \right)
\end{array} \right]
$$

$$
+\Delta t g \Delta t \sum_{k=bottom}^{surface} \left(\left(H_{ij+1k}^{t} + H_{ijk}^{t} \right) \frac{\sum_{k=i}^{k=surflayer} \left(\rho_{ij+1k} - \rho_{ijk} \right) \left(H_{ij+1k}^{t} + H_{ijk}^{t} \right) / 2}{\left(\rho_{ij+1k} + \rho_{ijk} \right) / 2\, \Delta x_{ij+\frac{1}{2}k}} \Bigg/ \left(2\Delta x_{ij} \right) \right)
$$

$$
-\Delta t \sum_{k=bottom}^{surface} \left[\left(H_{ij+1k}^{t} + H_{ijk}^{t} \right) \Delta t \left(\frac{\nu_x}{\Delta x_{ij+1k}^{2}} \left(u_{ij+2k}^{t} - u_{ij+1k}^{t} \right) - \frac{\nu_x}{\Delta x_{ijk}^{2}} \left(u_{ij+1k}^{t} - u_{ijk}^{t} \right) \right) \Big/ \left(2\Delta x_{ij} \right) \right]
$$

$$
-\Delta t \sum_{k=bottom}^{surface} \left[\left(H_{ij+1k}^{t} + H_{ijk}^{t} \right) \Delta t \left(\frac{\nu_y}{\Delta y_{i+\frac{1}{2}j+1k}^{2}} \left(u_{i+1j+1k}^{t} - u_{ij+1k}^{t} \right) - \frac{\nu_y}{\Delta y_{i-\frac{1}{2}j+1k}^{2}} \left(u_{ij+1k}^{t} - u_{i-1j+1k}^{t} \right) \right) \Big/ \left(2\Delta x_{ij} \right) \right]
$$

$$
-\Delta t \frac{\left(H_{ij+1k}^{t} + H_{ijk}^{t} \right) \Delta t \tau_{j}^{vento}}{\rho \Delta z_{ij+1k}} \Big/ \left(2\Delta x_{ij} \right)
$$

$$
+\Delta t \sum_{k=bottom}^{surface} \left(\frac{FlowInput_{ijk} - FlowOutput_{ijk}}{\Delta x_{ijk} \Delta y_{ij+\frac{1}{2}k}} \right)
$$

$$
-\Delta t \sum_{k=bottom}^{surface} \frac{f v_{ij+1k}^{*}}{2\Delta x_{ij}} - \frac{\Delta t}{2\Delta y_{ij+1k}} \left[\sum_{k=bottom}^{surface} \left(v_{ij+1k}^{t} \left(H_{ij+1k}^{t} + H_{ijk}^{t} \right) \right) \right]
$$

$$
+\Delta t^{2} \left(Cf_{ij+1k} + Cf_{ijk} \right) \left| u_{ij+1k}^{t} \right| \frac{\left[Other\_terms \right]}{\left(1 + \dfrac{\Delta t}{\left(H_{ij+1k}^{t} + H_{ijk}^{t} \right)} \left(Cf_{ij+1k} + Cf_{ijk} \right) \left| u_{ij+1k}^{t} \right| \right)} \Bigg/ \left(2\Delta x_{ij} \right)
$$

(35)

At the downstream boundary

$$-\frac{g\Delta t^2}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\frac{\xi_{ij-1}^{t+1/2}}{\Delta x_{ij-1/2}}\right)$$

$$+g\Delta t^3\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u_{ijk}^{t}\right|\frac{\left(\frac{\xi_{ij-1}^{t+1/2}}{\Delta x_{ij-1/2k}}\right)}{\left(1+\frac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u_{ijk}^{t}\right|\right)}\Bigg/2\Delta x_{ij}$$

$$+\xi_{ij}^{t+1/2}+\frac{g\Delta t^2}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\frac{\xi_{ij}^{t+1/2}}{\Delta x_{ij-1/2}}\right)$$

$$-g\Delta t^3\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u_{ijk}^{t}\right|\frac{\left(\frac{\xi_{ij}^{t+1/2}}{\Delta x_{ij-1/2k}}\right)}{\left(1+\frac{\Delta t}{\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}\left(Cf_{ijk}+Cf_{ij-1k}\right)\left|u_{ijk}^{t}\right|\right)}\Bigg/\left(2\Delta x_{ij}\right)$$

$$=$$

$$
= \xi^{t}_{ij} + \Delta t \sum\nolimits_{k=bottom}^{surface} \left[u^{t}_{ijk}\ \left(H^{t}_{ijk} + H^{t}_{ij-1k} \right) / 2\Delta x_{ij} \right]
$$

$$
-\Delta t \sum\nolimits_{k=bottom}^{surface} \left[
\begin{array}{l}
\Delta t \left[\dfrac{H^{t}_{ijk}\left(u^{t}_{ij+\frac{1}{2}k} + \left|u^{t}_{ij+\frac{1}{2}k}\right|\right) u^{t}_{ijk} - H^{t}_{ij-1k}\left(u^{t}_{ij-\frac{1}{2}k} + \left|u^{t}_{ij-\frac{1}{2}k}\right|\right) u^{t}_{ij-1k}}{\Delta x_{ij-\frac{1}{2}k}} + \dfrac{H^{t}_{ijk}\left(u^{t}_{ij+\frac{1}{2}k} - \left|u^{t}_{ij+\frac{1}{2}k}\right|\right) u^{t}_{ij+1k} - H^{t}_{ij-1k}\left(u^{t}_{ij-\frac{1}{2}k} - \left|u^{t}_{ij-\frac{1}{2}k}\right|\right) u^{t}_{ijk}}{\Delta x_{ij-\frac{1}{2}k}} \right] \Big/ \left(2\Delta x_{ij}\right) \\
+\Delta t \left[\dfrac{H^{t}_{i+\frac{1}{2}j-\frac{1}{2}k}\left(v^{t}_{i+\frac{1}{2}j-\frac{1}{2}k} + \left|v^{t}_{i+\frac{1}{2}j-\frac{1}{2}k}\right|\right) u^{t}_{ijk} - H^{t}_{i-\frac{1}{2}j-\frac{1}{2}k}\left(v^{t}_{i-\frac{1}{2}j-\frac{1}{2}k} + \left|v^{t}_{i-\frac{1}{2}j-\frac{1}{2}k}\right|\right) u^{t}_{i-1jk}}{\Delta y_{i-\frac{1}{2}jk}} + \dfrac{H^{t}_{i+\frac{1}{2}j-\frac{1}{2}k}\left(v^{t}_{i+\frac{1}{2}j-\frac{1}{2}k} - \left|v^{t}_{i+\frac{1}{2}j-\frac{1}{2}k}\right|\right) u^{t}_{i+1jk} - H^{t}_{i-\frac{1}{2}j-\frac{1}{2}k}\left(v^{t}_{i-\frac{1}{2}j-\frac{1}{2}k} - \left|v^{t}_{i-\frac{1}{2}j-\frac{1}{2}k}\right|\right) u^{t}_{ijk}}{\Delta y_{i-\frac{1}{2}jk}} \right] \Big/ \left(2\Delta x_{ij}\right) \\
+\Delta t \left(H^{t}_{ijk} + H^{t}_{ij-1k} \right) \left[\dfrac{\left(w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k+1} + \left|w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k+1}\right|\right) u^{t}_{ijk} - \left(w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k} + \left|w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k}\right|\right) u^{t}_{ijk-1}}{2H^{t}_{ij-\frac{1}{2}k}} + \dfrac{\left(w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k+1} - \left|w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k+1}\right|\right) u^{t}_{ijk+1} - \left(w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k} - \left|w^{t-\frac{1}{2}}_{ij-\frac{1}{2}k}\right|\right) u^{t}_{ijk}}{2H^{t}_{ij-\frac{1}{2}k}} \right] \Big/ \left(2\Delta x_{ij}\right)
\end{array}
\right]
$$

$$
-\Delta t g \Delta t \sum\nolimits_{k=kbottom}^{surface} \left(\left(H^{t}_{ijk} + H^{t}_{ij-1k} \right) \frac{\sum\limits_{k=i}^{k=surflayer} \left(\rho_{ijk} - \rho_{ij-1k}\right)\left(H^{t}_{ijk} + H^{t}_{ij-1k}\right)/2}{\left(\rho_{ijk} + \rho_{ij-1k}\right)/2\, \Delta x_{ij-\frac{1}{2}k}} \Big/ \left(2\Delta x_{ij}\right) \right)
$$

$$
+\Delta t \sum\nolimits_{k=bottom}^{surface} \left[\left(H^{t}_{ijk} + H^{t}_{ij-1k} \right) \Delta t \left(\frac{\nu_{x}}{\Delta x^{2}_{ijk}} \left(u^{t}_{ij+1k} - u^{t}_{ijk} \right) - \frac{\nu_{x}}{\Delta x^{2}_{ij-1k}} \left(u^{t}_{ijk} - u^{t}_{ij-1k} \right) \right) \Big/ \left(2\Delta x_{ij}\right) \right]
$$

$$
+\Delta t \sum\nolimits_{k=bottom}^{surface} \left[\left(H^{t}_{ijk} + H^{t}_{ij-1k} \right) \Delta t \left(\frac{\nu_{y}}{\Delta y^{2}_{i+\frac{1}{2}jk}} \left(u^{t}_{i+1jk} - u^{t}_{ijk} \right) - \frac{\nu_{y}}{\Delta y^{2}_{i-\frac{1}{2}jk}} \left(u^{t}_{ijk} - u^{t}_{i-1jk} \right) \right) \Big/ \left(2\Delta x_{ij}\right) \right]
$$

$$
+\Delta t \frac{\left(H^{t}_{ijk} + H^{t}_{ij-1k} \right) \Delta t \tau^{vento}_{ij}}{\rho \Delta z_{ijk}} \Big/ \left(2\Delta x_{ij}\right)
$$

$$
+\Delta t \sum\nolimits_{k=bottom}^{surface} \frac{f v^{*}_{ijk}}{4\Delta x_{ij}} + \frac{\Delta t}{2\Delta y_{ij}} \left[\sum_{k=bottom}^{surface} \left(v^{t}_{ijk} \left(H^{t}_{ijk} + H^{t}_{i-1jk} \right) \right) \right]
$$

$$
+\Delta t^{2} \left(Cf_{ijk} + Cf_{ij-1k} \right) \left| u^{t}_{ijk} \right| \frac{\left[Other\_terms \right]}{\left(1 + \dfrac{\Delta t}{\left(H^{t}_{ijk} + H^{t}_{ij-1k} \right)} \left(Cf_{ijk} + Cf_{ij-1k} \right) \left| u^{t}_{ijk} \right| \right)} \Big/ \left(2\Delta x_{ij}\right)
$$

(36)

Equations 34, 35 and 36 are rearranged with elevations in *t+1/2* on the left in order to obtain a tridiagonal system (37), which is solved implicitly for the elevations. The *r* term for this tridiagonal system is the right side of the same equations.

$$a_{ij}\xi_{ij-1}^{t+\frac{1}{2}} = \left[-\frac{g\Delta t^2}{2\Delta x_{ij}\Delta x_{ij-\frac{1}{2}}} \sum_{k=bottom}^{surface} \left(H_{ijk}^t + H_{ij-1k}^t \right) + \frac{g\Delta t^3 \left(Cf_{ijk} + Cf_{ij-1k} \right) \left| u_{ijk}^t \right|}{2\Delta x_{ij}\Delta x_{ij-\frac{1}{2}k} \left(1 + \frac{\Delta t}{\left(H_{ijk}^t + H_{ij-1k}^t \right)} \left(Cf_{ijk} + Cf_{ij-1k} \right) \left| u_{ijk}^t \right| \right)} \right] \xi_{ij-1}^{t+\frac{1}{2}}$$

$$b_{ij}\xi_{ij}^{t+\frac{1}{2}} = \left[\begin{array}{l} 1 + g\Delta t^2 \left(\sum_{k=bottom}^{surface} \left(\frac{\left(H_{ijk}^t + H_{ij-1k}^t \right)}{2\Delta x_{ij}\Delta x_{ij-\frac{1}{2}}} \right) + \sum_{i=ibottom}^{isurface} \left(\frac{\left(H_{ij+1k}^t + H_{ijk}^t \right)}{2\Delta x_{ij}\Delta x_{ij+\frac{1}{2}}} \right) \right) \\ -\frac{g\Delta t^3 \left(Cf_{ijk} + Cf_{ij-1k} \right) \left| u_{ijk}^t \right|}{2\Delta x_{ij}\Delta x_{ij-\frac{1}{2}k} \left(1 + \frac{\Delta t}{\left(H_{ijk}^t + H_{ij-1k}^t \right)} \left(Cf_{ijk} + Cf_{ij-1k} \right) \left| u_{ijk}^t \right| \right)} \\ -\frac{g\Delta t^3 \left(Cf_{ij+1k} + Cf_{ijk} \right) \left| u_{ij+1k}^t \right|}{2\Delta x_{ij}\Delta x_{ij+\frac{1}{2}k} \left(1 + \frac{\Delta t}{\left(H_{ij+1k}^t + H_{ijk}^t \right)} \left(Cf_{ij+1k} + Cf_{ijk} \right) \left| u_{ij+1k}^t \right| \right)} \end{array} \right] \xi_{ij}^{t+\frac{1}{2}}$$

$$c_{ij}\xi_{ij+1}^{t+\frac{1}{2}} = \left[-\frac{g\Delta t^2}{2\Delta x_{ij}\Delta x_{ij+\frac{1}{2}}} \sum_{k=bottom}^{surface} \left(H_{ij+1k}^t + H_{ijk}^t \right) + \frac{g\Delta t^3 \left(Cf_{ij+1k} + Cf_{ijk} \right) \left| u_{ij+1k}^t \right|}{2\Delta x_{ij}\Delta x_{ij+\frac{1}{2}k} \left(1 + \frac{\Delta t}{\left(H_{ij+1k}^t + H_{ijk}^t \right)} \left(Cf_{ij+1k} + Cf_{ijk} \right) \left| u_{ij+1k}^t \right| \right)} \right] \xi_{ij+1}^{t+\frac{1}{2}}$$

(37)

Semi time step = 2

The rationale is exactly the same as for semi time step 1.

$$\xi_{ij}^{t+1/2} = \xi_{ij}^{t} + \Delta t \sum_{k=bottom}^{surface} \left(\frac{u_{ijk}^{t}\left(H_{ijk}^{t} + H_{ij-1k}^{t}\right)}{2\Delta x_{ijk}} - \frac{u_{ij+1k}^{t}\left(H_{ijk}^{t} + H_{ij+1k}^{t}\right)}{2\Delta x_{ijk}} \right)$$
$$+\Delta t \sum_{k=bottom}^{surface} \left(\frac{v_{ijk}^{t+1}\left(H_{ijk}^{t} + H_{i-1jk}^{t}\right)}{2\Delta y_{ijk}} - \frac{v_{i+1jk}^{t+1}\left(H_{ijk}^{t} + H_{i+1jk}^{t}\right)}{2\Delta y_{ijk}} \right) \qquad (38)$$
$$+\Delta t \sum_{k=bottom}^{surface} \left(\frac{FlowInput_{ijk} - FlowOutput_{ijk}}{\Delta x_{ijk} \Delta y_{ijk}} \right)$$

$$
\begin{aligned}
&\sum_{k=bottom}^{surface}\left[v_{ijk}^{t+1}\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\right]=\sum_{k=bottom}^{surface}\left[v_{ij}^{t}\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\right] \\
&-\sum_{k=bottom}^{surface}\left[\begin{array}{l}
\Delta t\left[\begin{array}{l}
\dfrac{H_{ijk}^{t}\left(v_{i+1/2jk}^{t}+\left|v_{i+1/2jk}^{t}\right|\right)v_{ijk}^{t}-H_{i-1jk}^{t}\left(v_{i-1/2jk}^{t}+\left|v_{i-1/2jk}^{t}\right|\right)v_{i-1jk}^{t}}{\Delta y_{i-1/2jk}}+ \\
\dfrac{H_{ijk}^{t}\left(v_{i+1/2jk}^{t}-\left|v_{i+1/2jk}^{t}\right|\right)v_{i+1jk}^{t}-H_{i-1jk}^{t}\left(v_{i-1/2jk}^{t}-\left|v_{i-1/2jk}^{t}\right|\right)v_{ijk}^{t}}{\Delta y_{i-1/2jk}}
\end{array}\right] \\
+\Delta t\left[\begin{array}{l}
\dfrac{H_{i-1/2j+1/2k}^{t}\left(u_{i-1/2j+1/2k}^{t}+\left|u_{i-1/2j+1/2k}^{t}\right|\right)v_{ijk}^{t}-H_{i-1/2j-1/2k}^{t}\left(u_{i-1/2j-1/2k}^{t}+\left|u_{i-1/2j-1/2k}^{t}\right|\right)v_{ij-1k}^{t}}{\Delta x_{ij-1/2k}}+ \\
\dfrac{H_{i+1/2j-1/2k}^{t}\left(u_{i+1/2j-1/2k}^{t}-\left|u_{i+1/2j-1/2k}^{t}\right|\right)v_{i+1jk}^{t}-H_{i-1/2j-1/2k}^{t}\left(u_{i-1/2j-1/2k}^{t}-\left|u_{i-1/2j-1/2k}^{t}\right|\right)v_{ijk}^{t}}{\Delta x_{ij-1/2k}}
\end{array}\right] \\
+\Delta t\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right) \\
\left[\begin{array}{l}
\dfrac{\left(w_{i-1/2jk+1}^{t-1/2}+\left|w_{i-1/2jk+1}^{t-1/2}\right|\right)v_{ijk}^{t}-\left(w_{i-1/2jk}^{t-1/2}+\left|w_{i-1/2jk}^{t-1/2}\right|\right)v_{ijk-1}^{t}}{2H_{i-1/2jk}^{t}}+ \\
\dfrac{\left(w_{i-1/2jk+1}^{t-1/2}-\left|w_{i-1/2jk+1}^{t-1/2}\right|\right)v_{ijk+1}^{t}-\left(w_{i-1/2jk}^{t-1/2}-\left|w_{i-1/2jk}^{t-1/2}\right|\right)v_{ijk}^{t}}{2H_{i-1/2jk}^{t}}
\end{array}\right]
\end{array}\right] \\
&-g\Delta t\sum_{k=bottom}^{surface}\left(\begin{array}{l}
\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\dfrac{\xi_{ij}^{t+1/2}-\xi_{i-1j}^{t+1/2}}{\Delta y_{i-1/2jk}} \\
+\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\dfrac{\sum\limits_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{i-1jk}\right)\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)/2}{\left(\rho_{ijk}+\rho_{i-1jk}\right)/2\,\Delta y_{i-1/2jk}}
\end{array}\right) \\
&+\sum_{k=bottom}^{surface}\left[\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\Delta t\left(\frac{\nu_x}{\Delta y_{ijk}^{2}}\left(v_{i+1jk}^{t}-v_{ijk}^{t}\right)-\frac{\nu_x}{\Delta y_{i-1jk}^{2}}\left(v_{ijk}^{t}-v_{i-1jk}^{t}\right)\right)\right] \\
&+\sum_{k=bottom}^{surface}\left[\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\Delta t\left(\frac{\nu_y}{\Delta x_{ij+1/2k}^{2}}\left(v_{ij+1k}^{t}-v_{ijk}^{t}\right)-\frac{\nu_y}{\Delta x_{ij-1/2k}^{2}}\left(v_{ijk}^{t}-v_{ij-1k}^{t}\right)\right)\right] \\
&+\frac{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\Delta t\,\tau_{ij}^{vento}}{\rho H_{i-1/2jk}^{t}}-\Delta t\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^{t}\right|v_{ijk}^{t+1}-\Delta t\sum_{k=bottom}^{surface} f\,u_{ijk}^{*}
\end{aligned}
$$

(39)

$$
\begin{aligned}
&u^{t+1}_{ijk}\left(1+\frac{\Delta t}{\left(H^{t}_{ijk}+H^{t}_{i-1jk}\right)}\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v^{t}_{ijk}\right|\right)=v^{t}_{ijk}\\
&-\frac{\Delta t}{\left(H^{t}_{ijk}+H^{t}_{i-1jk}\right)}\left[\begin{array}{l}\dfrac{H^{t}_{ijk}\left(v^{t}_{i+\frac{1}{2}jk}+\left|v^{t}_{i+\frac{1}{2}jk}\right|\right)v^{t}_{ijk}-H^{t}_{i-1jk}\left(v^{t}_{i-\frac{1}{2}jk}+\left|v^{t}_{i-\frac{1}{2}jk}\right|\right)v^{t}_{i-1jk}}{\Delta y_{i-\frac{1}{2}jk}}+\\ \dfrac{H^{t}_{ijk}\left(v^{t}_{i+\frac{1}{2}jk}-\left|v^{t}_{i+\frac{1}{2}jk}\right|\right)v^{t}_{i+1jk}-H^{t}_{i-1jk}\left(v^{t}_{i-\frac{1}{2}jk}-\left|v^{t}_{i-\frac{1}{2}jk}\right|\right)v^{t}_{ijk}}{\Delta y_{i-\frac{1}{2}jk}}\end{array}\right]\\
&-\frac{\Delta t}{\left(H^{t}_{ijk}+H^{t}_{i-1jk}\right)}\left[\begin{array}{l}\dfrac{H^{t}_{i-\frac{1}{2}j+\frac{1}{2}k}\left(u^{t}_{i-\frac{1}{2}j+\frac{1}{2}k}+\left|u^{t}_{i-\frac{1}{2}j+\frac{1}{2}k}\right|\right)v^{t}_{ijk}-H^{t}_{i-\frac{1}{2}j-\frac{1}{2}k}\left(u^{t}_{i-\frac{1}{2}j-\frac{1}{2}k}+\left|u^{t}_{i-\frac{1}{2}j-\frac{1}{2}k}\right|\right)v^{t}_{ij-1k}}{\Delta x_{ij-\frac{1}{2}k}}+\\ \dfrac{H^{t}_{i-\frac{1}{2}j+\frac{1}{2}k}\left(u^{t}_{i-\frac{1}{2}j+\frac{1}{2}k}-\left|u^{t}_{i-\frac{1}{2}j+\frac{1}{2}k}\right|\right)v^{t}_{ij+1k}-H^{t}_{i-\frac{1}{2}j-\frac{1}{2}k}\left(u^{t}_{i-\frac{1}{2}j-\frac{1}{2}k}-\left|u^{t}_{i-\frac{1}{2}j-\frac{1}{2}k}\right|\right)v^{t}_{ijk}}{\Delta x_{ij-\frac{1}{2}k}}\end{array}\right]\\
&-\Delta t\left[\begin{array}{l}\dfrac{\left(w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk+1}+\left|w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk+1}\right|\right)v^{t}_{ijk}-\left(w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk}+\left|w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk}\right|\right)v^{t}_{ijk-1}}{2H^{t}_{i-\frac{1}{2}jk}}+\\ \dfrac{\left(w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk+1}-\left|w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk+1}\right|\right)v^{t}_{ijk+1}-\left(w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk}-\left|w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk}\right|\right)v^{t}_{ijk}}{2H^{t}_{i-\frac{1}{2}jk}}\end{array}\right]\\
&-g\Delta t\left(\frac{\xi^{t+\frac{1}{2}}_{ij}-\xi^{t+\frac{1}{2}}_{i-1j}}{\Delta x_{ij-\frac{1}{2}k}}+\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{i-1jk}\right)\left(H^{t}_{ijk}+H^{t}_{i-1jk}\right)/2}{\left(\rho_{ijk}+\rho_{i-1jk}\right)/2\,\Delta y_{i-\frac{1}{2}jk}}\right)\\
&+\frac{\nu_{x}}{\Delta y^{2}_{ijk}}\left(v^{t}_{i+1jk}-v^{t}_{ijk}\right)\Delta t-\frac{\nu_{x}}{\Delta y^{2}_{i-1jk}}\left(v^{t}_{ijk}-v^{t}_{i-1jk}\right)\Delta t\\
&+\frac{\nu_{y}}{\Delta x^{2}_{ij+\frac{1}{2}k}}\left(v^{t}_{ij+1k}-v^{t}_{ijk}\right)\Delta t-\frac{\nu_{y}}{\Delta x^{2}_{ij-\frac{1}{2}k}}\left(v^{t}_{ijk}-v^{t}_{ij-1k}\right)\Delta t\\
&+\frac{\nu_{ijk+1}}{\Delta z^{2}_{i-\frac{1}{2}jk+\frac{1}{2}}}\left(v^{t+1}_{ijk+1}-v^{t+1}_{ijk}\right)\Delta t-f\,u^{*}_{ijk}\Delta t
\end{aligned}
$$

(40)

$$
v_{ijk}^{t+1}=\frac{\left[\begin{array}{l}
v_{ijk}^{t}-\frac{\Delta t}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left[\begin{array}{l}
\frac{H_{ijk}^{t}\left(v_{i+\frac{1}{2}jk}^{t}+\left|v_{i+\frac{1}{2}jk}^{t}\right|\right)v_{ijk}^{t}-H_{i-1jk}^{t}\left(v_{i-\frac{1}{2}jk}^{t}+\left|v_{i-\frac{1}{2}jk}^{t}\right|\right)v_{i-1jk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}+\\
\frac{H_{ijk}^{t}\left(v_{i+\frac{1}{2}jk}^{t}-\left|v_{i+\frac{1}{2}jk}^{t}\right|\right)v_{i+1jk}^{t}-H_{i-1jk}^{t}\left(v_{i-\frac{1}{2}jk}^{t}-\left|v_{i-\frac{1}{2}jk}^{t}\right|\right)v_{ijk}^{t}}{\Delta y_{i-\frac{1}{2}jk}}
\end{array}\right]\\
-\frac{\Delta t}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left[\begin{array}{l}
\frac{H_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}\left(u_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}+\left|u_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}\right|\right)v_{ijk}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(u_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}+\left|u_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)v_{ij-1k}^{t}}{\Delta x_{ij-\frac{1}{2}k}}+\\
\frac{H_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}\left(u_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}-\left|u_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}\right|\right)v_{ij+1k}^{t}-H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(u_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}-\left|u_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\right|\right)v_{ijk}^{t}}{\Delta x_{ij-\frac{1}{2}k}}
\end{array}\right]\\
-\Delta t\left[\begin{array}{l}
\frac{\left(w_{i-\frac{1}{2}jk+1}^{t-\frac{1}{2}}+\left|w_{i-\frac{1}{2}jk+1}^{t-\frac{1}{2}}\right|\right)v_{ijk}^{t}-\left(w_{i-\frac{1}{2}jk}^{t-\frac{1}{2}}+\left|w_{i-\frac{1}{2}jk}^{t-\frac{1}{2}}\right|\right)v_{ijk-1}^{t}}{2H_{i-\frac{1}{2}jk}^{t}}+\\
\frac{\left(w_{i-\frac{1}{2}jk+1}^{t-\frac{1}{2}}-\left|w_{i-\frac{1}{2}jk+1}^{t-\frac{1}{2}}\right|\right)v_{ijk+1}^{t}-\left(w_{i-\frac{1}{2}jk}^{t-\frac{1}{2}}-\left|w_{i-\frac{1}{2}jk}^{t-\frac{1}{2}}\right|\right)v_{ijk}^{t}}{2H_{i-\frac{1}{2}jk}^{t}}
\end{array}\right]\\
-g\Delta t\left(\frac{\xi_{ij}^{t+\frac{1}{2}}-\xi_{i-1j}^{t+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}jk}}+\frac{\sum_{k=0}^{k=surflayer}\left(\rho_{ijk}-\rho_{i-1jk}\right)\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)/2}{\left(\rho_{ijk}+\rho_{i-1jk}\right)/2\,\Delta y_{i-\frac{1}{2}jk}}\right)\\
+\frac{\nu_{x}}{\Delta y_{ijk}^{2}}\left(v_{i+1jk}^{t}-v_{ijk}^{t}\right)\Delta t-\frac{\nu_{x}}{\Delta y_{i-1jk}^{2}}\left(v_{ijk}^{t}-v_{i-1jk}^{t}\right)\Delta t\\
+\frac{\nu_{y}}{\Delta x_{ij+\frac{1}{2}k}^{2}}\left(v_{ij+1k}^{t}-v_{ijk}^{t}\right)\Delta t-\frac{\nu_{y}}{\Delta x_{ij-\frac{1}{2}k}^{2}}\left(v_{ijk}^{t}-v_{ij-1k}^{t}\right)\Delta t\\
+\frac{\nu_{ijk+1}}{\Delta z_{i-\frac{1}{2}jk+\frac{1}{2}}^{2}}\left(v_{ijk+1}^{t+1}-v_{ijk}^{t+1}\right)\Delta t-f\,u_{ijk}^{*}\Delta t
\end{array}\right]}{\left(1+\frac{\Delta t}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^{t}\right|\right)}
$$

(41)

$$
v_{ijk}^{t+1}=\frac{\left[-g\Delta t\left(\frac{\xi_{ij}^{t+\frac{1}{2}}-\xi_{i-1j}^{t+\frac{1}{2}}}{\Delta x_{i-\frac{1}{2}jk}}\right)+Other\_terms\right]}{\left(1+\frac{\Delta t}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^{t}\right|\right)}
$$

(42)

$$
\begin{aligned}
&\xi_{ij}^{t+1/2} + \Delta tg\Delta t \sum_{k=bottom}^{surface} \left(\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \frac{\xi_{ij}^{t+1/2} - \xi_{i-1j}^{t+1/2}}{\Delta y_{i-1/2\,j}} \Big/ \left(2\Delta y_{ij} \right) \right) \\
&-\Delta tg\Delta t \sum_{k=bottom}^{surface} \left(\left(H_{i+1jk}^{t} + H_{ijk}^{t} \right) \frac{\xi_{i+1j}^{t+1/2} - \xi_{ij}^{t+1/2}}{\Delta y_{i+1/2\,j}} \Big/ \left(2\Delta y_{ij} \right) \right) = \xi_{ij}^{t} + \Delta t \sum_{k=bottom}^{surface} \left[v_{ijk}^{t} \left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \big/ 2\Delta y_{ij} \right] \\
&-\Delta t \sum_{k=bottom}^{surface} \left[\begin{aligned}
&\Delta t \left[\begin{aligned}
&\frac{H_{ijk}^{t}\left(v_{i+1/2\,jk}^{t} + \left| v_{i+1/2\,jk}^{t} \right| \right) v_{ijk}^{t} - H_{i-1jk}^{t}\left(v_{i-1/2\,jk}^{t} + \left| v_{i-1/2\,jk}^{t} \right| \right) v_{i-1jk}^{t}}{\Delta y_{i-1/2\,jk}} + \\
&\frac{H_{ijk}^{t}\left(v_{i+1/2\,jk}^{t} - \left| v_{i+1/2\,jk}^{t} \right| \right) v_{i+1jk}^{t} - H_{i-1jk}^{t}\left(v_{i-1/2\,jk}^{t} - \left| v_{i-1/2\,jk}^{t} \right| \right) v_{ijk}^{t}}{\Delta y_{i-1/2\,jk}}
\end{aligned} \right] \Bigg/ \left(2\Delta y_{ij} \right) \\
&+\Delta t \left[\begin{aligned}
&\frac{H_{i-1/2\,j+1/2\,k}^{t}\left(u_{i-1/2\,j+1/2\,k}^{t} + \left| u_{i-1/2\,j+1/2\,k}^{t} \right| \right) v_{ijk}^{t} - H_{i-1/2\,j-1/2\,k}^{t}\left(u_{i-1/2\,j-1/2\,k}^{t} + \left| u_{i-1/2\,j-1/2\,k}^{t} \right| \right) v_{ij-1k}^{t}}{\Delta x_{ij-1/2\,k}} + \\
&\frac{H_{i-1/2\,j+1/2\,k}^{t}\left(u_{i-1/2\,j+1/2\,k}^{t} - \left| u_{i-1/2\,j+1/2\,k}^{t} \right| \right) v_{ij+1k}^{t} - H_{i-1/2\,j-1/2\,k}^{t}\left(u_{i-1/2\,j-1/2\,k}^{t} - \left| u_{i-1/2\,j-1/2\,k}^{t} \right| \right) v_{ijk}^{t}}{\Delta x_{ij-1/2\,k}}
\end{aligned} \right] \Bigg/ \left(2\Delta y_{ij} \right) \\
&+\Delta t \left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \left[\begin{aligned}
&\frac{\left(w_{i-1/2\,jk+1}^{t-1/2} + \left| w_{i-1/2\,jk+1}^{t-1/2} \right| \right) v_{ijk}^{t} - \left(w_{i-1/2\,jk}^{t-1/2} + \left| w_{i-1/2\,jk}^{t-1/2} \right| \right) v_{ijk-1}^{t}}{2H_{i-1/2\,jk}^{t}} + \\
&\frac{\left(w_{i-1/2\,jk+1}^{t-1/2} - \left| w_{i-1/2\,jk+1}^{t-1/2} \right| \right) v_{ijk+1}^{t} - \left(w_{i-1/2\,jk}^{t-1/2} - \left| w_{i-1/2\,jk}^{t-1/2} \right| \right) v_{ijk}^{t}}{2H_{i-1/2\,jk}^{t}}
\end{aligned} \right] \Bigg/ \left(2\Delta y_{ij} \right)
\end{aligned} \right] \\
&-\Delta tg\Delta t \sum_{k=kbottom}^{surface} \left(\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \frac{\sum_{k=i}^{k=surflayer} \left(\rho_{ijk} - \rho_{i-1jk} \right)\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \big/ 2}{\left(\rho_{ijk} + \rho_{i-1jk} \right) \big/ 2\Delta y_{i-1/2\,jk}} \Bigg/ \left(2\Delta y_{ij} \right) \right) \\
&+\Delta t \sum_{k=bottom}^{surface} \left[\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \Delta t \left(\frac{\nu_x}{\Delta y_{ijk}^{2}} \left(v_{i+1jk}^{t} - v_{ijk}^{t} \right) - \frac{\nu_x}{\Delta y_{i-1jk}^{2}} \left(v_{ijk}^{t} - v_{i-1jk}^{t} \right) \right) \Big/ \left(2\Delta y_{ij} \right) \right] \\
&+\Delta t \sum_{k=bottom}^{surface} \left[\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \Delta t \left(\frac{\nu_y}{\Delta x_{ij+1/2\,k}^{2}} \left(v_{ij+1k}^{t} - v_{ijk}^{t} \right) - \frac{\nu_y}{\Delta x_{ij-1/2\,k}^{2}} \left(v_{ijk}^{t} - v_{ij-1k}^{t} \right) \right) \Big/ \left(2\Delta y_{ij} \right) \right] \\
&-\Delta t\Delta t \left(Cf_{ijk} + Cf_{i-1jk} \right) \left| v_{ijk}^{t} \right| v_{ijk}^{t+1} \big/ \left(4\Delta y_{ij} \right) + \Delta t \frac{\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \Delta t \, \tau_{j}^{vento}}{\rho \Delta z_{ijk}} \Bigg/ \left(2\Delta y_{ij} \right) \\
&+\frac{\Delta t}{2\Delta x_{ij}} \left[\sum_{k=bottom}^{surface} \left(u_{ijk}^{t} \left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \right) \right] - \Delta t \sum_{k=bottom}^{surface} \frac{f u_{ijk}^{*}}{2\Delta y_{ij}}
\end{aligned}
$$

$$
\begin{aligned}
&-\Delta t\sum_{k=bottom}^{surface}\left[v^{t}_{i+1jk}\;\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)\big/2\Delta y_{ij}\right]\\
&+\Delta t\sum_{k=bottom}^{surface}\left[\begin{aligned}
&\Delta t\left[\begin{aligned}&\frac{H^{t}_{i+1jk}\left(v^{t}_{i+3/2jk}+\left|v^{t}_{i+3/2jk}\right|\right)v^{t}_{i+1jk}-H^{t}_{ijk}\left(v^{t}_{i+1/2jk}+\left|v^{t}_{i+1/2jk}\right|\right)v^{t}_{i+1jk}}{\Delta y_{i+1/2jk}}+\\
&\frac{H^{t}_{i+1jk}\left(v^{t}_{i+3/2jk}-\left|v^{t}_{i+3/2jk}\right|\right)v^{t}_{i+2jk}-H^{t}_{ijk}\left(v^{t}_{i+1/2jk}-\left|v^{t}_{i+1/2jk}\right|\right)v^{t}_{i+1jk}}{\Delta y_{i+1/2jk}}\end{aligned}\right]\Bigg/\left(2\Delta y_{ij}\right)\\
&+\Delta t\left[\begin{aligned}&\frac{H^{t}_{i+1/2j+1/2k}\left(u^{t}_{i+1/2j+1/2k}+\left|u^{t}_{i+1/2j+1/2k}\right|\right)v^{t}_{i+1jk}-H^{t}_{i+1/2j-1/2k}\left(u^{t}_{i+1/2j-1/2k}+\left|u^{t}_{i+1/2j-1/2k}\right|\right)v^{t}_{i+1j-1k}}{\Delta x_{i+1/2jk}}+\\
&\frac{H^{t}_{i+1/2j+1/2k}\left(u^{t}_{i+1/2j+1/2k}-\left|u^{t}_{i+1/2j+1/2k}\right|\right)v^{t}_{i+1j+1k}-H^{t}_{i+1/2j-1/2k}\left(u^{t}_{i+1/2j-1/2k}-\left|u^{t}_{i+1/2j-1/2k}\right|\right)v^{t}_{i+1jk}}{\Delta x_{i+1/2jk}}\end{aligned}\right]\Bigg/\left(2\Delta y_{ij}\right)\\
&+\Delta t\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)\left[\begin{aligned}&\frac{\left(w^{t-1/2}_{i+1/2jk+1}+\left|w^{t-1/2}_{i+1/2jk+1}\right|\right)v^{t}_{i+1jk}-\left(w^{t-1/2}_{i+1/2jk}+\left|w^{t-1/2}_{i+1/2jk}\right|\right)v^{t}_{i+1jk-1}}{2H^{t}_{i+1/2jk}}+\\
&\frac{\left(w^{t-1/2}_{i+1/2jk+1}-\left|w^{t-1/2}_{i+1/2jk+1}\right|\right)v^{t}_{i+1jk+1}-\left(w^{t-1/2}_{i+1/2jk}-\left|w^{t-1/2}_{i+1/2jk}\right|\right)v^{t}_{i+1jk}}{2H^{t}_{i+1/2jk}}\end{aligned}\right]\Bigg/\left(2\Delta y_{ij}\right)
\end{aligned}\right]\\
&+\Delta t g\Delta t\sum_{k=bottom}^{surface}\left(\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)\frac{\sum_{k=i}^{k=surflayer}\left(\rho_{i+1jk}-\rho_{ijk}\right)\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)/2}{\left(\rho_{i+1jk}+\rho_{ijk}\right)\big/2\,\Delta y_{i+1/2jk}}\Bigg/\left(2\Delta y_{ij}\right)\right)\\
&-\Delta t\sum_{k=bottom}^{surface}\left[\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)\Delta t\left(\frac{\nu_x}{\Delta y^{2}_{i+1jk}}\left(v^{t}_{i+2jk}-v^{t}_{i+1jk}\right)-\frac{\nu_x}{\Delta y^{2}_{ijk}}\left(v^{t}_{i+1jk}-v^{t}_{ijk}\right)\right)\Big/\left(2\Delta y_{ij}\right)\right]\\
&-\Delta t\sum_{k=bottom}^{surface}\left[\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)\Delta t\left(\frac{\nu_y}{\Delta x^{2}_{i+1j+1/2k}}\left(v^{t}_{i+1j+1k}-v^{t}_{i+1jk}\right)-\frac{\nu_y}{\Delta x^{2}_{i+1j-1/2k}}\left(v^{t}_{i+1jk}-v^{t}_{i+1j-1k}\right)\right)\Big/\left(2\Delta y_{ij}\right)\right]\\
&+\Delta t\;\Delta t\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v^{t}_{i+1jk}\right|v^{t+1}_{i+1jk}\big/\left(2\Delta y_{ij}\right)-\Delta t\frac{\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)\Delta t\,\tau^{vento}_{j}}{\rho\Delta z_{i+1jk}}\Big/\left(2\Delta y_{ij}\right)\\
&+\Delta t\sum_{k=bottom}^{surface}\left(\frac{FlowInput_{ijk}-FlowOutput_{ijk}}{\Delta x_{ijk}\Delta y_{ijk}}\right)\\
&-\frac{\Delta t}{2\Delta x_{i+1jk}}\left[\sum_{k=bottom}^{surface}\left(u^{t}_{i+1jk}\left(H^{t}_{i+1jk}+H^{t}_{i+1j-1k}\right)\right)\right]+\Delta t\sum_{k=bottom}^{surface}\frac{f\,u^{*}_{i+1jk}}{2\Delta y_{ij}}
\end{aligned}
$$

(43)

$$
\begin{aligned}
&\xi_{ij}^{t+1/2}+\Delta t g\Delta t\sum_{k=bottom}^{surface}\left(\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\frac{\xi_{ij}^{t+1/2}-\xi_{i-1j}^{t+1/2}}{\Delta y_{i-1/2\,j}}\Big/\left(2\Delta y_{ij}\right)\right)\\
&-\Delta t g\Delta t\sum_{k=bottom}^{surface}\left(\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)\frac{\xi_{i+1j}^{t+1/2}-\xi_{ij}^{t+1/2}}{\Delta y_{i+1/2\,j}}\Big/\left(2\Delta y_{ij}\right)\right)\\
&+\Delta t\Delta t\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^{t}\right|v_{ijk}^{t+1}\Big/\left(2\Delta y_{ij}\right)\\
&+\Delta t\Delta t\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v_{i+1jk}^{t}\right|v_{i+1jk}^{t+1}\Big/\left(2\Delta y_{ij}\right)=\xi_{ij}^{t}+\Delta t\sum_{k=bottom}^{surface}\left[v_{ijk}^{t}\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\Big/2\Delta y_{ij}\right]\\
&-\Delta t\sum_{k=bottom}^{surface}\left[\begin{array}{l}
\Delta t\left[\begin{array}{l}\dfrac{H_{ijk}^{t}\left(v_{i+1/2jk}^{t}+\left|v_{i+1/2jk}^{t}\right|\right)v_{ijk}^{t}-H_{i-1jk}^{t}\left(v_{i-1/2jk}^{t}+\left|v_{i-1/2jk}^{t}\right|\right)v_{i-1jk}^{t}}{\Delta y_{i-1/2jk}}+\\[2ex]
\dfrac{H_{ijk}^{t}\left(v_{i+1/2jk}^{t}-\left|v_{i+1/2jk}^{t}\right|\right)v_{i+1jk}^{t}-H_{i-1jk}^{t}\left(v_{i-1/2jk}^{t}-\left|v_{i-1/2jk}^{t}\right|\right)v_{ijk}^{t}}{\Delta y_{i-1/2jk}}\end{array}\right]\Big/\left(2\Delta y_{ij}\right)\\[4ex]
+\Delta t\left[\begin{array}{l}\dfrac{H_{i-1/2j+1/2k}^{t}\left(u_{i-1/2j+1/2k}^{t}+\left|u_{i-1/2j+1/2k}^{t}\right|\right)v_{ijk}^{t}-H_{i-1/2j-1/2k}^{t}\left(u_{i-1/2j-1/2k}^{t}+\left|u_{i-1/2j-1/2k}^{t}\right|\right)v_{ij-1k}^{t}}{\Delta x_{ij-1/2k}}+\\[2ex]
\dfrac{H_{i-1/2j+1/2k}^{t}\left(u_{i-1/2j+1/2k}^{t}-\left|u_{i-1/2j+1/2k}^{t}\right|\right)v_{ij+1k}^{t}-H_{i-1/2j-1/2k}^{t}\left(u_{i-1/2j-1/2k}^{t}-\left|u_{i-1/2j-1/2k}^{t}\right|\right)v_{ijk}^{t}}{\Delta x_{ij-1/2k}}\end{array}\right]\Big/\left(2\Delta y_{ij}\right)\\[4ex]
+\Delta t\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\left[\begin{array}{l}\dfrac{\left(w_{i-1/2jk+1}^{t-1/2}+\left|w_{i-1/2jk+1}^{t-1/2}\right|\right)v_{ijk}^{t}-\left(w_{i-1/2jk}^{t-1/2}+\left|w_{i-1/2jk}^{t-1/2}\right|\right)v_{ijk-1}^{t}}{2H_{i-1/2jk}^{t}}+\\[2ex]
\dfrac{\left(w_{i-1/2jk+1}^{t-1/2}-\left|w_{i-1/2jk+1}^{t-1/2}\right|\right)v_{ijk+1}^{t}-\left(w_{i-1/2jk}^{t-1/2}-\left|w_{i-1/2jk}^{t-1/2}\right|\right)v_{ijk}^{t}}{2H_{i-1/2jk}^{t}}\end{array}\right]\Big/\left(2\Delta y_{ij}\right)
\end{array}\right]\\
&-\Delta t g\Delta t\sum_{k=kbottom}^{surface}\left(\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\frac{\sum_{k=i}^{k=surflayer}\left(\rho_{ijk}-\rho_{i-1jk}\right)\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\Big/2}{\left(\rho_{ijk}+\rho_{i-1jk}\right)\Big/2\,\Delta y_{i-1/2jk}}\Big/\left(2\Delta y_{ij}\right)\right)\\
&+\Delta t\sum_{k=bottom}^{surface}\left[\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\Delta t\left(\frac{\nu_{x}}{\Delta y_{ijk}^{2}}\left(v_{i+1jk}^{t}-v_{ijk}^{t}\right)-\frac{\nu_{x}}{\Delta y_{i-1jk}^{2}}\left(v_{ijk}^{t}-v_{i-1jk}^{t}\right)\right)\Big/\left(2\Delta y_{ij}\right)\right]\\
&+\Delta t\sum_{k=bottom}^{surface}\left[\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\Delta t\left(\frac{\nu_{y}}{\Delta x_{ij+1/2k}^{2}}\left(v_{ij+1k}^{t}-v_{ijk}^{t}\right)-\frac{\nu_{y}}{\Delta x_{ij-1/2k}^{2}}\left(v_{ijk}^{t}-v_{ij-1k}^{t}\right)\right)\Big/\left(2\Delta y_{ij}\right)\right]\\
&+\Delta t\frac{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\Delta t\,\tau_{j}^{vento}}{\rho\Delta z_{ijk}}\Big/\left(2\Delta y_{ij}\right)\\
&+\frac{\Delta t}{2\Delta x_{ij}}\left[\sum_{k=bottom}^{surface}\left(u_{ijk}^{t}\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\right)\right]-\Delta t\sum_{k=bottom}^{surface}\frac{f\,u_{ijk}^{*}}{2\Delta y_{ij}}
\end{aligned}
$$

$$
\begin{aligned}
&-\Delta t\sum_{k=bottom}^{surface}\left[v_{i+1jk}^{t}\;\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)\Big/2\Delta y_{ij}\right]\\
&+\Delta t\sum_{k=bottom}^{surface}\left[\begin{array}{l}
\Delta t\left[\begin{array}{l}\dfrac{H_{i+1jk}^{t}\left(v_{i+3/2jk}^{t}+\left|v_{i+3/2jk}^{t}\right|\right)v_{i+1jk}^{t}-H_{ijk}^{t}\left(v_{i+1/2jk}^{t}+\left|v_{i+1/2jk}^{t}\right|\right)v_{i+1jk}^{t}}{\Delta y_{i+1/2jk}}+\\ \dfrac{H_{i+1jk}^{t}\left(v_{i+3/2jk}^{t}-\left|v_{i+3/2jk}^{t}\right|\right)v_{i+2jk}^{t}-H_{ijk}^{t}\left(v_{i+1/2jk}^{t}-\left|v_{i+1/2jk}^{t}\right|\right)v_{i+1jk}^{t}}{\Delta y_{i+1/2jk}}\end{array}\right]\Bigg/\left(2\Delta y_{ij}\right)\\
+\Delta t\left[\begin{array}{l}\dfrac{H_{i+1/2j+1/2k}^{t}\left(u_{i+1/2j+1/2k}^{t}+\left|u_{i+1/2j+1/2k}^{t}\right|\right)v_{i+1jk}^{t}-H_{i+1/2j-1/2k}^{t}\left(u_{i+1/2j-1/2k}^{t}+\left|u_{i+1/2j-1/2k}^{t}\right|\right)v_{i+1j-1k}^{t}}{\Delta x_{i+1/2jk}}+\\ \dfrac{H_{i+1/2j+1/2k}^{t}\left(u_{i+1/2j+1/2k}^{t}-\left|u_{i+1/2j+1/2k}^{t}\right|\right)v_{i+1j+1k}^{t}-H_{i+1/2j-1/2k}^{t}\left(u_{i+1/2j-1/2k}^{t}-\left|u_{i+1/2j-1/2k}^{t}\right|\right)v_{i+1jk}^{t}}{\Delta x_{i+1/2jk}}\end{array}\right]\Bigg/\left(2\Delta y_{ij}\right)\\
+\Delta t\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)\left[\begin{array}{l}\dfrac{\left(w_{i+1/2jk+1}^{t-1/2}+\left|w_{i+1/2jk+1}^{t-1/2}\right|\right)v_{i+1jk}^{t}-\left(w_{i+1/2jk}^{t-1/2}+\left|w_{i+1/2jk}^{t-1/2}\right|\right)v_{i+1jk-1}^{t}}{2H_{i+1/2jk}^{t}}+\\ \dfrac{\left(w_{i+1/2jk+1}^{t-1/2}-\left|w_{i+1/2jk+1}^{t-1/2}\right|\right)v_{i+1jk+1}^{t}-\left(w_{i+1/2jk}^{t-1/2}-\left|w_{i+1/2jk}^{t-1/2}\right|\right)v_{i+1jk}^{t}}{2H_{i+1/2jk}^{t}}\end{array}\right]\Bigg/\left(2\Delta y_{ij}\right)
\end{array}\right]\\
&+\Delta tg\Delta t\sum_{k=bottom}^{surface}\left(\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)\frac{\sum_{k=i}^{k=surflayer}\left(\rho_{i+1jk}-\rho_{ijk}\right)\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)/2}{\left(\rho_{i+1jk}+\rho_{ijk}\right)\Big/2\,\Delta y_{i+1/2jk}}\Bigg/\left(2\Delta y_{ij}\right)\right)\\
&-\Delta t\sum_{k=bottom}^{surface}\left[\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)\Delta t\left(\frac{\nu_{x}}{\Delta y_{i+1jk}^{2}}\left(v_{i+2jk}^{t}-v_{i+1jk}^{t}\right)-\frac{\nu_{x}}{\Delta y_{ijk}^{2}}\left(v_{i+1jk}^{t}-v_{ijk}^{t}\right)\right)\Big/\left(2\Delta y_{ij}\right)\right]\\
&-\Delta t\sum_{k=bottom}^{surface}\left[\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)\Delta t\left(\frac{\nu_{y}}{\Delta x_{i+1j+1/2k}^{2}}\left(v_{i+1j+1k}^{t}-v_{i+1jk}^{t}\right)-\frac{\nu_{y}}{\Delta x_{i+1j-1/2k}^{2}}\left(v_{i+1jk}^{t}-v_{i+1j-1k}^{t}\right)\right)\Big/\left(2\Delta y_{ij}\right)\right]\\
&-\Delta t\frac{\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)\Delta t\,\tau_{j}^{vento}}{\rho\Delta z_{i+1jk}}\Bigg/\left(2\Delta y_{ij}\right)\\
&+\Delta t\sum_{k=bottom}^{surface}\left(\frac{FlowInput_{ijk}-FlowOutput_{ijk}}{\Delta x_{ijk}\Delta y_{ijk}}\right)\\
&-\frac{\Delta t}{2\Delta x_{i+1jk}}\left[\sum_{k=bottom}^{surface}\left(u_{i+1jk}^{t}\left(H_{i+1jk}^{t}+H_{i+1j-1k}^{t}\right)\right)\right]+\Delta t\sum_{k=bottom}^{surface}\frac{f\,u_{i+1jk}^{*}}{2\Delta y_{ij}}
\end{aligned}
$$

(44)

$$
\begin{aligned}
&\xi_{ij}^{t+\frac{1}{2}} + \Delta t g \Delta t \sum_{k=bottom}^{surface} \left(\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \frac{\xi_{ij}^{t+\frac{1}{2}} - \xi_{i-1j}^{t+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}j}} \Big/ \left(2\Delta y_{ij} \right) \right) \\
&- \Delta t g \Delta t \sum_{k=bottom}^{surface} \left(\left(H_{i+1jk}^{t} + H_{ijk}^{t} \right) \frac{\xi_{i+1j}^{t+\frac{1}{2}} - \xi_{ij}^{t+\frac{1}{2}}}{\Delta y_{i+\frac{1}{2}j}} \Big/ \left(2\Delta y_{ij} \right) \right) \\
&+ \Delta t \Delta t \left(Cf_{ijk} + Cf_{i-1jk} \right) \left| v_{ijk}^{t} \right| \frac{\left[-g\Delta t \left(\frac{\xi_{ij}^{t+\frac{1}{2}} - \xi_{i-1j}^{t+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}jk}} \right) + Other\_terms \right]}{\left(1 + \frac{\Delta t}{\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right)} \left(Cf_{ijk} + Cf_{i-1jk} \right) \left| v_{ijk}^{t} \right| \right)} \Bigg/ \left(2\Delta y_{ij} \right) \\
&- \Delta t \Delta t \left(Cf_{i+1jk} + Cf_{ijk} \right) \left| v_{i+1jk}^{t} \right| \frac{\left[-g\Delta t \left(\frac{\xi_{i+1j}^{t+\frac{1}{2}} - \xi_{ij}^{t+\frac{1}{2}}}{\Delta y_{i+\frac{1}{2}jk}} \right) + Other\_terms \right]}{\left(1 + \frac{\Delta t}{\left(H_{i+1jk}^{t} + H_{ijk}^{t} \right)} \left(Cf_{i+1jk} + Cf_{ijk} \right) \left| v_{i+1jk}^{t} \right| \right)} \Bigg/ \left(2\Delta y_{ij} \right) \\
&= see\_above
\end{aligned}
$$

(45)

$$
-\frac{g\Delta t^2}{2\Delta y_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\frac{\xi_{i-1j}^{t+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}j}}\right)
$$

$$
+g\Delta t^3\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^{t}\right|\frac{\left(\frac{\xi_{i-1j}^{t+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}jk}}\right)}{\left(1+\frac{\Delta t}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^{t}\right|\right)}\Bigg/ 2\Delta y_{ij}
$$

$$
+\xi_{ij}^{t+\frac{1}{2}}+\frac{g\Delta t^2}{2\Delta y_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\frac{\xi_{ij}^{t+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}j}}\right)+\frac{g\Delta t^2}{2\Delta y_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)\frac{\xi_{ij}^{t+\frac{1}{2}}}{\Delta y_{i+\frac{1}{2}j}}\right)
$$

$$
-g\Delta t^3\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^{t}\right|\frac{\left(\frac{\xi_{ij}^{t+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}jk}}\right)}{\left(1+\frac{\Delta t}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^{t}\right|\right)}\Bigg/\left(2\Delta y_{ij}\right)
$$

$$
-g\Delta t^3\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v_{i+1jk}^{t}\right|\frac{\left(\frac{\xi_{ij}^{t+\frac{1}{2}}}{\Delta y_{i+\frac{1}{2}jk}}\right)}{\left(1+\frac{\Delta t}{\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)}\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v_{i+1jk}^{t}\right|\right)}\Bigg/\left(2\Delta y_{ij}\right)
$$

$$
-\frac{g\Delta t^2}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)\frac{\xi_{i+1j}^{t+\frac{1}{2}}}{\Delta y_{i+\frac{1}{2}j}}\right)
$$

$$
+g\Delta t^3\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v_{i+1jk}^{t}\right|\frac{\left(\frac{\xi_{i+1j}^{t+\frac{1}{2}}}{\Delta y_{i+\frac{1}{2}jk}}\right)}{\left(1+\frac{\Delta t}{\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)}\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v_{i+1jk}^{t}\right|\right)}\Bigg/\left(2\Delta y_{ij}\right)
$$

$$
= see\_above
$$

$$
-\Delta t^2\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^{t}\right|\frac{\left[Other\_terms\right]}{\left(1+\frac{\Delta t}{\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^{t}\right|\right)}\Bigg/\left(2\Delta y_{ij}\right)
$$

$$
+\Delta t^2\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v_{i+1jk}^{t}\right|\frac{\left[Other\_terms\right]}{\left(1+\frac{\Delta t}{\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)}\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v_{i+1jk}^{t}\right|\right)}\Bigg/\left(2\Delta y_{ij}\right)
$$

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$$
-\frac{g\Delta t^2}{2\Delta y_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{ijk}^t+H_{i-1jk}^t\right)\frac{\xi_{i-1j}^{t+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}j}}\right)
$$

$$
+g\Delta t^3\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^t\right|\frac{\left(\frac{\xi_{i-1j}^{t+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}jk}}\right)}{\left(1+\frac{\Delta t}{\left(H_{ijk}^t+H_{i-1jk}^t\right)}\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^t\right|\right)}\Bigg/ 2\Delta y_{ij}
$$

$$
+\xi_{ij}^{t+\frac{1}{2}}+\frac{g\Delta t^2}{2\Delta y_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{ijk}^t+H_{i-1jk}^t\right)\frac{\xi_{ij}^{t+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}j}}\right)+\frac{g\Delta t^2}{2\Delta y_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{i+1jk}^t+H_{ijk}^t\right)\frac{\xi_{ij}^{t+\frac{1}{2}}}{\Delta y_{i+\frac{1}{2}j}}\right)
$$

$$
-g\Delta t^3\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^t\right|\frac{\left(\frac{\xi_{ij}^{t+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}jk}}\right)}{\left(1+\frac{\Delta t}{\left(H_{ijk}^t+H_{i-1jk}^t\right)}\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^t\right|\right)}\Bigg/\left(2\Delta y_{ij}\right)
$$

$$
-g\Delta t^3\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v_{i+1jk}^t\right|\frac{\left(\frac{\xi_{ij}^{t+\frac{1}{2}}}{\Delta y_{i+\frac{1}{2}jk}}\right)}{\left(1+\frac{\Delta t}{\left(H_{i+1jk}^t+H_{ijk}^t\right)}\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v_{i+1jk}^t\right|\right)}\Bigg/\left(2\Delta y_{ij}\right)
$$

$$
-\frac{g\Delta t^2}{2\Delta x_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{i+1jk}^t+H_{ijk}^t\right)\frac{\xi_{i+1j}^{t+\frac{1}{2}}}{\Delta y_{i+\frac{1}{2}j}}\right)
$$

$$
+g\Delta t^3\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v_{i+1jk}^t\right|\frac{\left(\frac{\xi_{i+1j}^{t+\frac{1}{2}}}{\Delta y_{i+\frac{1}{2}jk}}\right)}{\left(1+\frac{\Delta t}{\left(H_{i+1jk}^t+H_{ijk}^t\right)}\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v_{i+1jk}^t\right|\right)}\Bigg/\left(2\Delta y_{ij}\right)=
$$

$$
\begin{aligned}
&\xi_{ij}^{t} + \Delta t \sum_{k=bottom}^{surface} \left[v_{ijk}^{t} \left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \big/ 2\Delta y_{ij} \right] \\
&-\Delta t \sum_{k=bottom}^{surface} \left[\begin{aligned}
&\Delta t \left[\begin{aligned} &\frac{H_{ijk}^{t}\left(v_{i+\frac{1}{2}jk}^{t} + \left| v_{i+\frac{1}{2}jk}^{t} \right| \right) v_{ijk}^{t} - H_{i-1jk}^{t}\left(v_{i-\frac{1}{2}jk}^{t} + \left| v_{i-\frac{1}{2}jk}^{t} \right| \right) v_{i-1jk}^{t}}{\Delta y_{i-\frac{1}{2}jk}} + \\ &\frac{H_{ijk}^{t}\left(v_{i+\frac{1}{2}jk}^{t} - \left| v_{i+\frac{1}{2}jk}^{t} \right| \right) v_{i+1jk}^{t} - H_{i-1jk}^{t}\left(v_{i-\frac{1}{2}jk}^{t} - \left| v_{i-\frac{1}{2}jk}^{t} \right| \right) v_{ijk}^{t}}{\Delta y_{i-\frac{1}{2}jk}} \end{aligned} \right] \Bigg/ \left(2\Delta y_{ij} \right) \\
&+\Delta t \left[\begin{aligned} &\frac{H_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}\left(u_{i-\frac{1}{2}j+\frac{1}{2}k}^{t} + \left| u_{i-\frac{1}{2}j+\frac{1}{2}k}^{t} \right| \right) v_{ijk}^{t} - H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(u_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} + \left| u_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} \right| \right) v_{ij-1k}^{t}}{\Delta x_{ij-\frac{1}{2}k}} + \\ &\frac{H_{i-\frac{1}{2}j+\frac{1}{2}k}^{t}\left(u_{i-\frac{1}{2}j+\frac{1}{2}k}^{t} - \left| u_{i-\frac{1}{2}j+\frac{1}{2}k}^{t} \right| \right) v_{ij+1k}^{t} - H_{i-\frac{1}{2}j-\frac{1}{2}k}^{t}\left(u_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} - \left| u_{i-\frac{1}{2}j-\frac{1}{2}k}^{t} \right| \right) v_{ijk}^{t}}{\Delta x_{ij-\frac{1}{2}k}} \end{aligned} \right] \Bigg/ \left(2\Delta y_{ij} \right) \\
&+\Delta t \left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \left[\begin{aligned} &\frac{\left(w_{i-\frac{1}{2}jk+1}^{t-\frac{1}{2}} + \left| w_{i-\frac{1}{2}jk+1}^{t-\frac{1}{2}} \right| \right) v_{ijk}^{t} - \left(w_{i-\frac{1}{2}jk}^{t-\frac{1}{2}} + \left| w_{i-\frac{1}{2}jk}^{t-\frac{1}{2}} \right| \right) v_{ijk-1}^{t}}{2H_{i-\frac{1}{2}jk}^{t}} + \\ &\frac{\left(w_{i-\frac{1}{2}jk+1}^{t-\frac{1}{2}} - \left| w_{i-\frac{1}{2}jk+1}^{t-\frac{1}{2}} \right| \right) v_{ijk+1}^{t} - \left(w_{i-\frac{1}{2}jk}^{t-\frac{1}{2}} - \left| w_{i-\frac{1}{2}jk}^{t-\frac{1}{2}} \right| \right) v_{ijk}^{t}}{2H_{i-\frac{1}{2}jk}^{t}} \end{aligned} \right] \Bigg/ \left(2\Delta y_{ij} \right)
\end{aligned} \right] \\
&-g\Delta t^{2} \sum_{k=bottom}^{surface} \left(\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \frac{\sum_{k=i}^{k=surflayer} \left(\rho_{ijk} - \rho_{i-1jk} \right)\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \big/ 2}{\left(\rho_{ijk} + \rho_{i-1jk} \right) \big/ 2\,\Delta y_{i-\frac{1}{2}jk}} \Bigg/ \left(2\Delta y_{ij} \right) \right) \\
&+\Delta t^{2} \sum_{k=bottom}^{surface} \left[\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \left(\frac{\nu_{x}}{\Delta y_{ijk}^{2}} \left(v_{i+1jk}^{t} - v_{ijk}^{t} \right) - \frac{\nu_{x}}{\Delta y_{i-1jk}^{2}} \left(v_{ijk}^{t} - v_{i-1jk}^{t} \right) \right) \Big/ \left(2\Delta y_{ij} \right) \right] \\
&+\Delta t^{2} \sum_{k=bottom}^{surface} \left[\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \left(\frac{\nu_{y}}{\Delta x_{ij+\frac{1}{2}k}^{2}} \left(v_{ij+1k}^{t} - v_{ijk}^{t} \right) - \frac{\nu_{y}}{\Delta x_{ij-\frac{1}{2}k}^{2}} \left(v_{ijk}^{t} - v_{ij-1k}^{t} \right) \right) \Big/ \left(2\Delta y_{ij} \right) \right] \\
&+\Delta t \frac{\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right) \Delta t \tau_{j}^{vento}}{\rho \Delta z_{ijk}} \Big/ \left(2\Delta y_{ij} \right) \\
&+\frac{\Delta t}{2\Delta x_{ijk}} \left[\sum_{k=bottom}^{surface} \left(u_{ijk}^{t} \left(H_{ijk}^{t} + H_{ij-1k}^{t} \right) \right) \right] - \Delta t \sum_{k=bottom}^{surface} \frac{f u_{ijk}^{*}}{2\Delta y_{ij}} \\
&-\Delta t^{2} \left(Cf_{ijk} + Cf_{i-1jk} \right) \left| v_{ijk}^{t} \right| \frac{\left[Other\_terms \right]}{\left(1 + \frac{\Delta t}{\left(H_{ijk}^{t} + H_{i-1jk}^{t} \right)} \left(Cf_{ijk} + Cf_{i-1jk} \right) \left| v_{ijk}^{t} \right| \right)} \Bigg/ \left(2\Delta y_{ij} \right)
\end{aligned}
$$

$$
\begin{aligned}
&-\Delta t\sum_{k=bottom}^{surface}\left[v^{t}_{i+1jk}\ \left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)/2\Delta y_{ij}\right]\\
&+\Delta t\sum_{k=bottom}^{surface}\left[\begin{aligned}
&\Delta t\left[\begin{aligned}&\frac{H^{t}_{i+1jk}\left(v^{t}_{i+3/2jk}+\left|v^{t}_{i+3/2jk}\right|\right)v^{t}_{i+1jk}-H^{t}_{ijk}\left(v^{t}_{i+1/2jk}+\left|v^{t}_{i+1/2jk}\right|\right)v^{t}_{i+1jk}}{\Delta y_{i+1/2jk}}+\\
&\frac{H^{t}_{i+1jk}\left(v^{t}_{i+3/2jk}-\left|v^{t}_{i+3/2jk}\right|\right)v^{t}_{i+2jk}-H^{t}_{ijk}\left(v^{t}_{i+1/2jk}-\left|v^{t}_{i+1/2jk}\right|\right)v^{t}_{i+1jk}}{\Delta y_{i+1/2jk}}\end{aligned}\right]\Big/\left(2\Delta y_{ij}\right)\\
&+\Delta t\left[\begin{aligned}&\frac{H^{t}_{i+1/2j+1/2k}\left(u^{t}_{i+1/2j+1/2k}+\left|u^{t}_{i+1/2j+1/2k}\right|\right)v^{t}_{i+1jk}-H^{t}_{i+1/2j-1/2k}\left(u^{t}_{i+1/2j-1/2k}+\left|u^{t}_{i+1/2j-1/2k}\right|\right)v^{t}_{i+1j-1k}}{\Delta x_{i+1/2jk}}+\\
&\frac{H^{t}_{i+1/2j+1/2k}\left(u^{t}_{i+1/2j+1/2k}-\left|u^{t}_{i+1/2j+1/2k}\right|\right)v^{t}_{i+1j+1k}-H^{t}_{i+1/2j-1/2k}\left(u^{t}_{i+1/2j-1/2k}-\left|u^{t}_{i+1/2j-1/2k}\right|\right)v^{t}_{i+1jk}}{\Delta x_{i+1/2jk}}\end{aligned}\right]\Big/\left(2\Delta y_{ij}\right)\\
&+\Delta t\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)\left[\begin{aligned}&\frac{\left(w^{t-1/2}_{i+1/2jk+1}+\left|w^{t-1/2}_{i+1/2jk+1}\right|\right)v^{t}_{i+1jk}-\left(w^{t-1/2}_{i+1/2jk}+\left|w^{t-1/2}_{i+1/2jk}\right|\right)v^{t}_{i+1jk-1}}{2H^{t}_{i+1/2jk}}+\\
&\frac{\left(w^{t-1/2}_{i+1/2jk+1}-\left|w^{t-1/2}_{i+1/2jk+1}\right|\right)v^{t}_{i+1jk+1}-\left(w^{t-1/2}_{i+1/2jk}-\left|w^{t-1/2}_{i+1/2jk}\right|\right)v^{t}_{i+1jk}}{2H^{t}_{i+1/2jk}}\end{aligned}\right]\Big/\left(2\Delta y_{ij}\right)
\end{aligned}\right]\\
&+\Delta tg\Delta t\sum_{k=bottom}^{surface}\left(\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)\frac{\sum_{k=i}^{k=surflayer}\left(\rho_{i+1jk}-\rho_{ijk}\right)\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)/2}{\left(\rho_{i+1jk}+\rho_{ijk}\right)/2\Delta y_{i+1/2jk}}\Big/\left(2\Delta y_{ij}\right)\right)\\
&-\Delta t\sum_{k=bottom}^{surface}\left[\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)\Delta t\ \left(\frac{\nu_x}{\Delta y^2_{i+1jk}}\left(v^{t}_{i+2jk}-v^{t}_{i+1jk}\right)-\frac{\nu_x}{\Delta y^2_{ijk}}\left(v^{t}_{i+1jk}-v^{t}_{ijk}\right)\right)\Big/\left(2\Delta y_{ij}\right)\right]\\
&-\Delta t\sum_{k=bottom}^{surface}\left[\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)\Delta t\left(\frac{\nu_y}{\Delta x^2_{i+1j+1/2k}}\left(v^{t}_{i+1j+1k}-v^{t}_{i+1jk}\right)-\frac{\nu_y}{\Delta x^2_{i+1j-1/2k}}\left(v^{t}_{i+1jk}-v^{t}_{i+1j-1k}\right)\right)\Big/\left(2\Delta y_{ij}\right)\right]\\
&-\Delta t\frac{\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)\Delta t\,\tau^{vento}_{j}}{\rho\Delta z_{i+1jk}}\Big/\left(2\Delta y_{ij}\right)+\Delta t\sum_{k=bottom}^{surface}\left(\frac{FlowInput_{ijk}-FlowOutput_{ijk}}{\Delta x_{ijk}\Delta y_{ijk}}\right)\\
&-\frac{\Delta t}{2\Delta x_{i+1jk}}\left[\sum_{k=bottom}^{surface}\left(u^{t}_{i+1jk}\left(H^{t}_{i+1jk}+H^{t}_{i+1j-1k}\right)\right)\right]+\Delta t\sum_{k=bottom}^{surface}\frac{f\,u^{*}_{i+1jk}}{2\Delta y_{ij}}\\
&+\Delta t^2\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v^{t}_{i+1jk}\right|\frac{\left[Other\_terms\right]}{\left(1+\frac{\Delta t}{\left(H^{t}_{i+1jk}+H^{t}_{ijk}\right)}\left(Cf_{i+1jk}+Cf_{ijk}\right)\left|v^{t}_{i+1jk}\right|\right)}\Big/\left(2\Delta y_{ij}\right)
\end{aligned}
\tag{47}
$$

At the upstream boundary:

$$\xi_{ij}^{t+1/2} + \frac{g\Delta t^2}{2\Delta y_{ij}} \sum_{k=bottom}^{surface} \left(\left(H_{i+1\,jk}^{t} + H_{ijk}^{t} \right) \frac{\xi_{ij}^{t+1/2}}{\Delta y_{i+1/2\,j}} \right)$$

$$-g\Delta t^3 \left(Cf_{\,i+1\,jk} + Cf_{\,ijk} \right) \left| v_{i+1\,jk}^{t} \right| \frac{\left(\dfrac{\xi_{ij}^{t+1/2}}{\Delta y_{i+1/2\,jk}} \right)}{\left(1 + \dfrac{\Delta t}{\left(H_{i+1\,jk}^{t} + H_{ijk}^{t} \right)} \left(Cf_{\,i+1\,jk} + Cf_{\,ijk} \right) \left| v_{i+1\,jk}^{t} \right| \right)} \Bigg/ \left(2\Delta y_{ij} \right)$$

$$-\frac{g\Delta t^2}{2\Delta x_{ij}} \sum_{k=bottom}^{surface} \left(\left(H_{i+1\,jk}^{t} + H_{ijk}^{t} \right) \frac{\xi_{i+1\,j}^{t+1/2}}{\Delta y_{i+1/2\,j}} \right)$$

$$+g\Delta t^3 \left(Cf_{\,i+1\,jk} + Cf_{\,ijk} \right) \left| v_{i+1\,jk}^{t} \right| \frac{\left(\dfrac{\xi_{i+1\,j}^{t+1/2}}{\Delta y_{i+1/2\,jk}} \right)}{\left(1 + \dfrac{\Delta t}{\left(H_{i+1\,jk}^{t} + H_{ijk}^{t} \right)} \left(Cf_{\,i+1\,jk} + Cf_{\,ijk} \right) \left| v_{i+1\,jk}^{t} \right| \right)} \Bigg/ \left(2\Delta y_{ij} \right) =$$

$$
\begin{aligned}
&\xi_{ij}^{t} + \Delta t \sum_{k=bottom}^{surface} \left(\frac{v_{riverjk}^{t+1} H_{ij}^{t} \Delta x_{riverjk}}{2\Delta y_{ijk} \Delta x_{ij+1/2k}} \right) \\
&+\Delta t \sum_{k=bottom}^{surface} \left(\frac{v_{riverjk}^{t+1} H_{ij}^{t} \Delta x_{riverjk}}{2\Delta y_{ijk} \Delta x_{ij+1/2k}} \right) - \Delta t \sum_{k=bottom}^{surface} \left[v_{i+1jk}^{t} \left(H_{i+1jk}^{t} + H_{ijk}^{t} \right) \Big/ \left(2\Delta y_{ij} \right) \right] \\
&+\Delta t \sum_{k=bottom}^{surface} \left[\begin{array}{l}
\Delta t \left[\begin{array}{l}
\dfrac{H_{i+1jk}^{t} \left(v_{i+3/2jk}^{t} + \left| v_{i+3/2jk}^{t} \right| \right) v_{i+1jk}^{t} - H_{ijk}^{t} \left(v_{i+1/2jk}^{t} + \left| v_{i+1/2jk}^{t} \right| \right) v_{ijk}^{t}}{\Delta y_{i+1/2jk}} + \\
\dfrac{H_{i+1jk}^{t} \left(v_{i+3/2jk}^{t} - \left| v_{i+3/2jk}^{t} \right| \right) v_{i+2jk}^{t} - H_{ijk}^{t} \left(v_{i+1/2jk}^{t} - \left| v_{i+1/2jk}^{t} \right| \right) v_{i+1jk}^{t}}{\Delta y_{i+1/2jk}}
\end{array} \right] \Bigg/ \left(2\Delta y_{ij} \right) \\
+\Delta t \left[\begin{array}{l}
\dfrac{H_{i+1/2j+1/2k}^{t} \left(u_{i+1/2j+1/2k}^{t} + \left| u_{i+1/2j+1/2k}^{t} \right| \right) v_{i+1jk}^{t} - H_{i+1/2j-1/2k}^{t} \left(u_{i+1/2j-1/2k}^{t} + \left| u_{i+1/2j-1/2k}^{t} \right| \right) v_{i+1j-1k}^{t}}{\Delta x_{i+1/2jk}} + \\
\dfrac{H_{i+1/2j+1/21k}^{t} \left(u_{i+1/2j+1/2k}^{t} - \left| u_{i+1/2j+1/2k}^{t} \right| \right) v_{i+1j+1k}^{t} - H_{i+1/2j-1/2k}^{t} \left(u_{i+1/2j-1/2k}^{t} - \left| u_{i+1/2j-1/2k}^{t} \right| \right) v_{i+1jk}^{t}}{\Delta x_{i+1/2jk}}
\end{array} \right] \Bigg/ \left(2\Delta y_{ij} \right) \\
+\Delta t \left(H_{i+1jk}^{t} + H_{ijk}^{t} \right) \left[\begin{array}{l}
\dfrac{\left(w_{i+1/2jk+1}^{t-1/2} + \left| w_{i+1/2jk+1}^{t-1/2} \right| \right) v_{i+1jk}^{t} - \left(w_{i+1/2jk}^{t-1/2} + \left| w_{i+1/2jk}^{t-1/2} \right| \right) v_{i+1jk-1}^{t}}{2H_{i+1/2jk}^{t}} + \\
\dfrac{\left(w_{i+1/2jk+1}^{t-1/2} - \left| w_{i+1/2jk+1}^{t-1/2} \right| \right) v_{i+1jk+1}^{t} - \left(w_{i+1/2jk}^{t-1/2} - \left| w_{i+1/2jk}^{t-1/2} \right| \right) v_{i+1jk}^{t}}{2H_{i+1/2jk}^{t}}
\end{array} \right] \Bigg/ \left(2\Delta y_{ij} \right)
\end{array} \right] \\
&+\Delta t g \Delta t \sum_{k=bottom}^{surface} \left(\left(H_{i+1jk}^{t} + H_{ijk}^{t} \right) \frac{\sum_{k=i}^{k=surflayer} \left(\rho_{i+1jk} - \rho_{ijk} \right) \left(H_{i+1jk}^{t} + H_{ijk}^{t} \right) \big/ 2}{\left(\rho_{i+1jk} + \rho_{ijk} \right) \big/ 2\,\Delta y_{i+1/2jk}} \Bigg/ \left(2\Delta y_{ij} \right) \right) \\
&-\Delta t \sum_{k=bottom}^{surface} \left[\left(H_{i+1jk}^{t} + H_{ijk}^{t} \right) \Delta t \left(\frac{\nu_x}{\Delta y_{i+1jk}^{2}} \left(v_{i+2jk}^{t} - v_{i+1jk}^{t} \right) - \frac{\nu_x}{\Delta y_{ijk}^{2}} \left(v_{i+1jk}^{t} - v_{ijk}^{t} \right) \right) \Big/ \left(2\Delta y_{ij} \right) \right] \\
&-\Delta t \sum_{k=bottom}^{surface} \left[\left(H_{i+1jk}^{t} + H_{ijk}^{t} \right) \Delta t \left(\frac{\nu_y}{\Delta x_{i+1j+1/2k}^{2}} \left(v_{i+1j+1k}^{t} - v_{i+1jk}^{t} \right) - \frac{\nu_y}{\Delta x_{i+1j-1/2k}^{2}} \left(v_{i+1jk}^{t} - v_{i+1j-1k}^{t} \right) \right) \Big/ \left(2\Delta y_{ij} \right) \right] \\
&-\Delta t \frac{\left(H_{i+1jk}^{t} + H_{ijk}^{t} \right) \Delta t \tau_{j}^{vento}}{\rho \Delta z_{i+1jk}} \Bigg/ \left(2\Delta y_{ij} \right) + \Delta t \sum_{k=bottom}^{surface} \left(\frac{FlowInput_{ijk} - FlowOutput_{ijk}}{\Delta x_{ijk} \Delta y_{i+1/2jk}} \right) \\
&-\frac{\Delta t}{2\Delta y_{ij+1k}} \left[\sum_{k=bottom}^{surface} \left(u_{i+1jk}^{t} \left(H_{i+1jk}^{t} + H_{ijk}^{t} \right) \right) \right] + \Delta t \sum_{k=bottom}^{surface} \frac{f u_{i+1jk}^{*}}{2\Delta y_{ij}} \\
&+\Delta t^{2} \left(Cf_{i+1jk} + Cf_{ijk} \right) \left| v_{i+1jk}^{t} \right| \frac{\left[Other\_terms \right]}{\left(1 + \dfrac{\Delta t}{\left(H_{i+1jk}^{t} + H_{ijk}^{t} \right)} \left(Cf_{i+1jk} + Cf_{ijk} \right) \left| v_{i+1jk}^{t} \right| \right)} \Bigg/ \left(2\Delta y_{ij} \right)
\end{aligned}
$$

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At the downstream boundary

$$
-\frac{g\Delta t^2}{2\Delta y_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{ijk}^t+H_{i-1jk}^t\right)\frac{\xi_{i-1j}^{t+1/2}}{\Delta y_{i-1/2j}}\right)
$$

$$
+g\Delta t^3\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^t\right|\frac{\left(\dfrac{\xi_{i-1j}^{t+1/2}}{\Delta y_{i-1/2jk}}\right)}{\left(1+\dfrac{\Delta t}{\left(H_{ijk}^t+H_{i-1jk}^t\right)}\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^t\right|\right)}\Bigg/ 2\Delta y_{ij}
$$

$$
+\xi_{ij}^{t+1/2}+\frac{g\Delta t^2}{2\Delta y_{ij}}\sum_{k=bottom}^{surface}\left(\left(H_{ijk}^t+H_{i-1jk}^t\right)\frac{\xi_{ij}^{t+1/2}}{\Delta y_{i-1/2j}}\right)
$$

$$
-g\Delta t^3\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^t\right|\frac{\left(\dfrac{\xi_{ij}^{t+1/2}}{\Delta y_{i-1/2jk}}\right)}{\left(1+\dfrac{\Delta t}{\left(H_{ijk}^t+H_{i-1jk}^t\right)}\left(Cf_{ijk}+Cf_{i-1jk}\right)\left|v_{ijk}^t\right|\right)}\Bigg/\left(2\Delta y_{ij}\right)=
$$

$$
\begin{aligned}
&\xi_{ij}^{t+\frac{1}{2}} + \Delta t g \Delta t \sum_{k=bottom}^{surface} \left(\left(H^{t}_{ijk} + H^{t}_{i-1jk} \right) \frac{\xi_{ij}^{t+\frac{1}{2}} - \xi_{i-1j}^{t+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}j}} \Big/ \left(2\Delta y_{ij} \right) \right) \\
&= \xi^{t}_{ij} + \Delta t \sum_{k=bottom}^{surface} \left[v^{t}_{ijk} \left(H^{t}_{ijk} + H^{t}_{i-1jk} \right) / 2\Delta y_{ij} \right] \\
&-\Delta t \sum_{k=bottom}^{surface} \left[\begin{array}{l}
\Delta t \left[\dfrac{H^{t}_{ijk}\left(v^{t}_{i+\frac{1}{2}jk} + \left| v^{t}_{i+\frac{1}{2}jk} \right| \right) v^{t}_{ijk} - H^{t}_{i-1jk}\left(v^{t}_{i-\frac{1}{2}jk} + \left| v^{t}_{i-\frac{1}{2}jk} \right| \right) v^{t}_{i-1jk}}{\Delta y_{i-\frac{1}{2}jk}} + \dfrac{H^{t}_{ijk}\left(v^{t}_{i+\frac{1}{2}jk} - \left| v^{t}_{i+\frac{1}{2}jk} \right| \right) v^{t}_{i+1jk} - H^{t}_{i-1jk}\left(v^{t}_{i-\frac{1}{2}jk} - \left| v^{t}_{i-\frac{1}{2}jk} \right| \right) v^{t}_{ijk}}{\Delta y_{ij-\frac{1}{2}k}} \right] \Big/ \left(2\Delta y_{ij} \right) \\
+\Delta t \left[\dfrac{H^{t}_{i-\frac{1}{2}j+\frac{1}{2}k}\left(u^{t}_{i-\frac{1}{2}j+\frac{1}{2}k} + \left| u^{t}_{i-\frac{1}{2}j+\frac{1}{2}k} \right| \right) v^{t}_{ijk} - H^{t}_{i-\frac{1}{2}j-\frac{1}{2}k}\left(u^{t}_{i-\frac{1}{2}j-\frac{1}{2}k} + \left| u^{t}_{i-\frac{1}{2}j-\frac{1}{2}k} \right| \right) v^{t}_{ij-1k}}{\Delta x_{i-\frac{1}{2}jk}} + \dfrac{H^{t}_{i-\frac{1}{2}j+\frac{1}{2}k}\left(u^{t}_{i-\frac{1}{2}j+\frac{1}{2}k} - \left| u^{t}_{i-\frac{1}{2}j+\frac{1}{2}k} \right| \right) v^{t}_{ij+1k} - H^{t}_{i-\frac{1}{2}j-\frac{1}{2}k}\left(u^{t}_{i-\frac{1}{2}j-\frac{1}{2}k} - \left| u^{t}_{i-\frac{1}{2}j-\frac{1}{2}k} \right| \right) v^{t}_{ijk}}{\Delta x_{i-\frac{1}{2}jk}} \right] \Big/ \left(2\Delta y_{ij} \right) \\
+\Delta t \left(H^{t}_{ijk} + H^{t}_{i-1jk} \right) \left[\dfrac{\left(w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk+1} + \left| w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk+1} \right| \right) v^{t}_{ijk} - \left(w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk} + \left| w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk} \right| \right) v^{t}_{ijk-1}}{2H^{t}_{i-\frac{1}{2}jk}} + \dfrac{\left(w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk+1} - \left| w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk+1} \right| \right) v^{t}_{ijk+1} - \left(w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk} - \left| w^{t-\frac{1}{2}}_{i-\frac{1}{2}jk} \right| \right) v^{t}_{ijk}}{2H^{t}_{i-\frac{1}{2}jk}} \right] \Big/ \left(2\Delta y_{ij} \right)
\end{array} \right] \\
&-\Delta t g \Delta t \sum_{k=kbottom}^{surface} \left(\left(H^{t}_{ijk} + H^{t}_{i-1jk} \right) \frac{\sum\limits_{k=i}^{k=surflayer} \left(\rho_{ijk} - \rho_{i-1jk} \right) \left(H^{t}_{ijk} + H^{t}_{i-1jk} \right) / 2}{\left(\rho_{ijk} + \rho_{i-1jk} \right) / 2\, \Delta y_{i-\frac{1}{2}jk}} \Big/ \left(2\Delta y_{ij} \right) \right) \\
&+\Delta t \sum_{k=bottom}^{surface} \left[\left(H^{t}_{ijk} + H^{t}_{i-1jk} \right) \Delta t \left(\frac{\nu_x}{\Delta y^{2}_{ijk}} \left(v^{t}_{i+1jk} - v^{t}_{ijk} \right) - \frac{\nu_x}{\Delta y^{2}_{i-1jk}} \left(v^{t}_{ijk} - v^{t}_{i-1jk} \right) \right) \Big/ \left(2\Delta y_{ij} \right) \right] \\
&+\Delta t \sum_{k=bottom}^{surface} \left[\left(H^{t}_{ijk} + H^{t}_{i-1jk} \right) \Delta t \left(\frac{\nu_y}{\Delta x^{2}_{ij+\frac{1}{2}k}} \left(v^{t}_{ij+1k} - v^{t}_{ijk} \right) - \frac{\nu_y}{\Delta x^{2}_{ij-\frac{1}{2}k}} \left(v^{t}_{ijk} - v^{t}_{ij-1k} \right) \right) \Big/ \left(2\Delta y_{ij} \right) \right] \\
&+\Delta t \frac{\Delta y_{ijk} \left(H^{t}_{ijk} + H^{t}_{i-1jk} \right) \Delta t \, \tau^{vento}_{j}}{\rho \Delta z_{ijk}} \Big/ \left(2\Delta y_{ij} \right) + \frac{\Delta t}{2\Delta x_{ijk}} \left[\sum_{k=bottom}^{surface} \left(u^{t}_{ijk} \left(H^{t}_{ijk} + H^{t}_{ij-1k} \right) \right) \right] + \Delta t \sum_{k=bottom}^{surface} \frac{f u^{*}_{ijk}}{2\Delta y_{ij}} \\
&-\Delta t^{2} \left(Cf_{ijk} + Cf_{i-1jk} \right) \left| v^{t}_{ijk} \right| \frac{[Other\_terms]}{\left(1 + \dfrac{\Delta t}{\left(H^{t}_{ijk} + H^{t}_{i-1jk} \right)} \left(Cf_{ijk} + Cf_{i-1jk} \right) \left| v^{t}_{ijk} \right| \right)} \Big/ \left(2\Delta y_{ij} \right) \\
&-\Delta t \sum_{k=bottom}^{surface} \left(\frac{v^{t+1}_{iseak} H^{t}_{ijk} \Delta y_{iseak}}{2\Delta x_{ijk} \Delta y_{i+\frac{1}{2}jk}} \right) - \Delta t \sum_{k=bottom}^{surface} \left(\frac{v^{t}_{iseak} H^{t}_{ij} \Delta y_{iseak}}{2\Delta x_{ijk} \Delta y_{i+\frac{1}{2}jk}} \right) + \Delta t \sum_{k=bottom}^{surface} \left(\frac{FlowInput_{ijk} - FlowOutput_{ijk}}{\Delta x_{ijk} \Delta y_{i+\frac{1}{2}jk}} \right)
\end{aligned}
$$

(49)

$$
\begin{aligned}
a_{ij}\xi_{ij-1}^{t+1/2} &= -\xi_{i-1j}^{t+1/2}\frac{g\Delta t^2}{2\Delta y_{ij}\Delta y_{i-1/2j}}\sum_{k=bottom}^{surface}\left(H_{ijk}^t + H_{ij-1k}^t\right) \\
&+ g\Delta t^3\left(Cf_{ijk} + Cf_{i-1jk}\right)\left|v_{ijk}^t\right|\frac{\left(\dfrac{\xi_{i-1j}^{t+1/2}}{\Delta y_{i-1/2jk}}\right)}{\left(1+\dfrac{\Delta t}{\left(H_{ijk}^t + H_{i-1jk}^t\right)}\left(Cf_{ijk} + Cf_{i-1jk}\right)\left|v_{ijk}^t\right|\right)}\Bigg/ 2\Delta y_{ij} \\
b_{ij}\xi_{ij}^{t+1/2} &= \xi_{ij}^{t+1/2}\left[1 + g\Delta t^2\left(\sum_{k=bottom}^{surface}\left(\frac{\left(H_{ijk}^t + H_{i-1jk}^t\right)}{2\Delta y_{ij}\Delta y_{i-1/2j}}\right) + \sum_{i=ibottom}^{isurface}\left(\frac{\left(H_{i+1jk}^t + H_{ijk}^t\right)}{2\Delta y_{ij}\Delta y_{i+1/2j}}\right)\right)\right] \\
&- g\Delta t^3\left(Cf_{ijk} + Cf_{i-1jk}\right)\left|v_{ijk}^t\right|\frac{\left(\dfrac{\xi_{ij}^{t+1/2}}{\Delta y_{i-1/2jk}}\right)}{\left(1+\dfrac{\Delta t}{\left(H_{ijk}^t + H_{i-1jk}^t\right)}\left(Cf_{ijk} + Cf_{i-1jk}\right)\left|v_{ijk}^t\right|\right)}\Bigg/\left(2\Delta y_{ij}\right) \\
&- g\Delta t^3\left(Cf_{i+1jk} + Cf_{ijk}\right)\left|v_{i+1jk}^t\right|\frac{\left(\dfrac{\xi_{ij}^{t+1/2}}{\Delta y_{i+1/2jk}}\right)}{\left(1+\dfrac{\Delta t}{\left(H_{i+1jk}^t + H_{ijk}^t\right)}\left(Cf_{i+1jk} + Cf_{ijk}\right)\left|v_{i+1jk}^t\right|\right)}\Bigg/\left(2\Delta y_{ij}\right) \qquad (50) \\
c_{ij}\xi_{ij+1}^{t+1/2} &= -\xi_{i+1j}^{t+1/2}\frac{g\Delta t^2}{2\Delta y_{ij}\Delta y_{i+1/2j}}\sum_{k=bottom}^{surface}\left(H_{i+1jk}^t + H_{ijk}^t\right) \\
&+ g\Delta t^3\left(Cf_{i+1jk} + Cf_{ijk}\right)\left|v_{i+1jk}^t\right|\frac{\left(\dfrac{\xi_{i+1j}^{t+1/2}}{\Delta y_{i+1/2jk}}\right)}{\left(1+\dfrac{\Delta t}{\left(H_{i+1jk}^t + H_{ijk}^t\right)}\left(Cf_{i+1jk} + Cf_{ijk}\right)\left|v_{i+1jk}^t\right|\right)}\Bigg/\left(2\Delta y_{ij}\right)
\end{aligned}
$$

3.2.2 Vertical velocity calculation

The elevations calculated at *t+1/2* are used to compute the *u* velocities at *t+1* and the *w* velocities at *t+1/2*. Vertical velocities are calculated assuming volume conservation. Knowing the horizontal inflows, *w* may be calculated by difference, starting at the bottom layer (numerical solution of 4):

$$\begin{aligned}
& w_{ijk+1}^{t+1/2}\Delta y_{ijk}\Delta x_{ijk} - w_{ijk}^{t}\Delta y_{ijk}\Delta x_{ijk} = -\frac{u_{ij+1k}^{*}\left(H_{ijk}^{t}+H_{ij+1k}^{t}\right)\Delta y_{ij+1k} - u_{ijk}^{*}\left(H_{ij-1k}^{t}+H_{ijk}^{t}\right)\Delta y_{ijk}}{2} \\
& -\frac{v_{i+1jk}^{*}\left(H_{ijk}^{t}+H_{i+1jk}^{t}\right)\Delta x_{i+1jk} - v_{ijk}^{*}\left(H_{i-1jk}^{t}+H_{ijk}^{t}\right)\Delta x_{ijk}}{2} \\
& +FlowInput_{ijk} - FlowOutput_{ijk}
\end{aligned} \quad (51)$$

$$\begin{aligned}
& w_{ijk+1}^{t+1/2} = w_{ijk}^{t} - \frac{u_{ij+1k}^{*}\left(H_{ijk}^{t}+H_{ij+1k}^{t}\right) - u_{ijk}^{*}\left(H_{ij-1k}^{t}+H_{ijk}^{t}\right)}{2\Delta x_{ijk}} \\
& -\frac{v_{i+1jk}^{*}\left(H_{ijk}^{t}+H_{i+1jk}^{t}\right) - v_{ijk}^{*}\left(H_{i-1jk}^{t}+H_{ijk}^{t}\right)}{2\Delta y_{ijk}} \\
& +\left(\frac{FlowInput_{ijk} - FlowOutput_{ijk}}{\Delta y_{ijk}\Delta x_{ijk}}\right)
\end{aligned} \quad (52)$$

The asterisk means that the velocity component estimates to be used should be the same used previously in the continuity equation. Volume changes at surface grid cells may be calculated from flow divergence (equation 53) and its numerical solution (54):

$$\frac{dV_{ijk}}{dt} = \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\Delta z\Delta x\Delta y \quad (53)$$

$$\begin{aligned}
& V_{Surfj}^{t+1} = V_{ijk}^{t} + \frac{\left(u_{ij+1k}^{t}+u_{ij+1k}^{t+1}\right)\left(H_{ijk}^{t}+H_{ij+1k}^{t}\right)\Delta y_{ij+1k} - \left(u_{ijk}^{t}+u_{ijk}^{t+1}\right)\left(H_{ij-1k}^{t}+H_{ijk}^{t}\right)\Delta y_{ij}}{4}\Delta t \\
& +\frac{\left(v_{i+1jk}^{t}+v_{i+1jk}^{t+1}\right)\left(H_{ijk}^{t}+H_{i+1jk}^{t}\right)\Delta x_{i+1jk} - \left(v_{ijk}^{t}+v_{ijk}^{t+1}\right)\left(H_{i-1jk}^{t}+H_{ijk}^{t}\right)\Delta x_{ijk}}{4}\Delta t \\
& +w_{ijk}^{t}\Delta x_{ijk}\Delta y_{ij+1/2k-1/2}\Delta t \\
& +\left(FlowInput_{ijk} - FlowOutput_{ijk}\right)\Delta t
\end{aligned} \quad (54)$$

An updated estimate of elevation is then obtained from cell volume, width and length.

3.3 Transport equation

Finally, the concentration of any property may be calculated with the transport equation (5) using its numerical solution (AII.30). Where *S* is the concentration of any property and the terms *Ax and Az* stand for horizontal and vertical diffusivity, respectively. In the present model, horizontal diffusivity of momentum or any other property are assumed to be constant. Vertical diffusivity of momentum is calculated from a mixing length model as described in Santos (1995). Vertical diffusivity of mass and heat is calculated from Leendertse and Liu (1978) fide Santos (1995).

$$
\begin{aligned}
&\frac{S_{ijk}^{t+1}H_{ijk}^{t+1}\Delta x_{ijk}\Delta y_{ijk}-S_{ijk}^{t}H_{ijk}^{t}\Delta x_{ijk}\Delta y_{ijk}}{\Delta t}=\\
&-\left(\frac{u_{ij+1k}^{t*}\left(H_{ijk}^{t}+H_{ij+1k}^{t}\right)\Delta y_{ijk}}{2}S_{i*k}^{t}-\frac{u_{ijk}^{t*}\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\Delta y_{ijk}}{2}S_{i*k}^{t}\right)\\
&-\left(\frac{v_{i+1jk}^{t*}\left(H_{ijk}^{t}+H_{i+1jk}^{t}\right)\Delta x_{ijk}}{2}S_{*jk}^{t}-\frac{v_{ijk}^{t*}\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\Delta x_{ijk}}{2}S_{*jk}^{t}\right)\\
&-\left(w_{ijk+1}^{t}\Delta x_{ijk}\Delta y_{ijk}S_{ij*}^{t}-w_{ijk}^{t}\Delta x_{ijk}\Delta y_{ijk}S_{ij*}^{t}\right)+\\
&A_{xij+1k}\frac{\left(S_{ij+1}^{t}-S_{ij}^{t}\right)\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\Delta y_{ijk}}{2\Delta x_{ij+1/2k}}-A_{xijk}\frac{\left(S_{ijk}^{t}-S_{ij-1k}^{t}\right)\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\Delta y_{ijk}}{2\Delta x_{ij-1/2k}}+\\
&A_{yi+1jk}\frac{\left(S_{i+1j}^{t}-S_{ij}^{t}\right)\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)\Delta x_{ijk}}{2\Delta y_{i+1/2jk}}-A_{yijk}\frac{\left(S_{ijk}^{t}-S_{i-1jk}^{t}\right)\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\Delta x_{ijk}}{2\Delta y_{i-1/2jk}}+\\
&A_{zijk+1}\frac{\left(S_{ijk+1}^{t+1}-S_{ijk}^{t+1}\right)\Delta y_{ijk}\Delta x_{ijk}}{H_{ijk+\frac{1}{2}}}-A_{zijk}\frac{\left(S_{ijk}^{t+1}-S_{ijk-1}^{t+1}\right)\Delta y_{ijk}\Delta x_{ijk}}{H_{ijk-\frac{1}{2}}}
\end{aligned}
$$

(55)

$$\frac{S_{ijk}^{t+1}H_{ijk}^{t+1}-S_{ijk}^{t}H_{ijk}^{t}}{\Delta t}=$$

$$-\left(\frac{u_{ij+1k}^{t*}\left(H_{ijk}^{t}+H_{ij+1k}^{t}\right)}{2\Delta x_{ijk}}S_{i*k}^{t}-\frac{u_{ijk}^{t*}\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}{2\Delta x_{ijk}}S_{i*k}^{t}\right)$$

$$-\left(\frac{v_{i+1jk}^{t*}\left(H_{ijk}^{t}+H_{i+1jk}^{t}\right)}{2\Delta y_{ijk}}S_{*jk}^{t}-\frac{v_{ijk}^{t*}\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}{2\Delta y_{ijk}}S_{*jk}^{t}\right)$$

$$-\left(w_{ijk+1}^{t}S_{ij*}^{t}-w_{ijk}^{t}S_{ij*}^{t}\right)+$$

$$A_{xij+1k}\frac{\left(S_{ij+1}^{t}-S_{ij}^{t}\right)\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)}{2\Delta x_{ij+1/2k}\Delta x_{ijk}}-A_{xijk}\frac{\left(S_{ijk}^{t}-S_{ij-1k}^{t}\right)\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}{2\Delta x_{ij-1/2k}\Delta x_{ijk}}+$$

$$A_{yi+1jk}\frac{\left(S_{i+1j}^{t}-S_{ij}^{t}\right)\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)}{2\Delta y_{i+1/2jk}\Delta y_{ijk}}-A_{yijk}\frac{\left(S_{ijk}^{t}-S_{i-1jk}^{t}\right)\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}{2\Delta y_{i-1/2jk}\Delta y_{ijk}}+$$

$$A_{zijk+1}\frac{\left(S_{ijk+1}^{t+1}-S_{ijk}^{t+1}\right)}{H_{ijk+1/2}}-A_{zijk}\frac{\left(S_{ijk}^{t+1}-S_{ijk-1}^{t+1}\right)}{H_{ijk-1/2}}$$

(56)

By rearranging 56 with *t+1* terms on the left, another implicit solution based on a tridiagonal system may be obtained (57, 58 and 59).

$$S_{ijk}^{t+1}H_{ijk}^{t+1}-A_{zijk+1}\frac{\left(S_{ijk+1}^{t+1}-S_{ijk}^{t+1}\right)\Delta t}{H_{ijk+1/2}}+A_{zijk}\frac{\left(S_{ijk}^{t+1}-S_{ijk-1}^{t+1}\right)\Delta t}{H_{ijk-1/2}}=S_{ijk}^{t}H_{ijk}^{t}$$

$$-\left(\frac{u_{ij+1k}^{t*}\left(H_{ijk}^{t}+H_{ij+1k}^{t}\right)}{2\Delta x_{ijk}}S_{i*k}^{t}-\frac{u_{ijk}^{t*}\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)}{2\Delta x_{ijk}}S_{i*k}^{t}\right)\Delta t$$

$$-\left(\frac{v_{i+1jk}^{t*}\left(H_{ijk}^{t}+H_{i+1jk}^{t}\right)}{2\Delta y_{ijk}}S_{*jk}^{t}-\frac{v_{ijk}^{t*}\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)}{2\Delta y_{ijk}}S_{*jk}^{t}\right)\Delta t$$

$$-\left(w_{ijk+1}^{t}S_{ij*}^{t}-w_{ijk}^{t}S_{ij*}^{t}\right)\Delta t+$$

$$A_{xij+1k}\frac{\left(S_{ij+1}^{t}-S_{ij}^{t}\right)\left(H_{ij+1k}^{t}+H_{ijk}^{t}\right)\Delta t}{2\Delta x_{ij+1/2k}\Delta x_{ijk}}-A_{xijk}\frac{\left(S_{ijk}^{t}-S_{ij-1k}^{t}\right)\left(H_{ijk}^{t}+H_{ij-1k}^{t}\right)\Delta t}{2\Delta x_{ij-1/2k}\Delta x_{ijk}}+$$

$$A_{yi+1jk}\frac{\left(S_{i+1j}^{t}-S_{ij}^{t}\right)\left(H_{i+1jk}^{t}+H_{ijk}^{t}\right)\Delta t}{2\Delta y_{i+1/2jk}\Delta y_{ijk}}-A_{yijk}\frac{\left(S_{ijk}^{t}-S_{i-1jk}^{t}\right)\left(H_{ijk}^{t}+H_{i-1jk}^{t}\right)\Delta t}{2\Delta y_{i-1/2jk}\Delta y_{ijk}}$$

(57)

$$-A_{zijk}\frac{S_{ijk-1}^{t+1}\Delta t}{H_{ijk-1/2}}+S_{ijk}^{t+1}H_{ijk}^{t+1}+A_{zijk}\frac{S_{ijk}^{t+1}\Delta t}{H_{ijk-1/2}}+A_{zijk+1}\frac{S_{ijk}^{t+1}\Delta t}{H_{ijk+1/2}}-A_{zijk+1}\frac{S_{ijk+1}^{t+1}\Delta t}{H_{ijk+1/2}}=X^{t}$$

(58)

Where X^t represents all the explicit terms on the right of 57.

$$a_{ijk}S_{ijk-1}^{t+1}+b_{ijk}S_{ijk}^{t+1}+c_{ijk}S_{i+1jk}^{t+1}=X^{t}$$

$$a_{ijk}=-\frac{Azijk\Delta t}{H\,ijk-1/2}$$

$$b_{ijk}=H_{ijk}^{t+1}+\frac{Azijk\Delta t}{H\,ijk-1/2}+\frac{Azijk+1\Delta t}{H\,ijk+1/2} \tag{59}$$

$$c_{ijk}=-\frac{Azijk+1\Delta t}{H\,ijk+1/2}$$

3.4 Sea boundary conditions

Current velocity across a western sea boundary (v component = 0):

Surface

$$\left[u_{ijk}^{t+1}H_{ijk}^{t+1}-\frac{u_{ij+1k}^{t+*}\left(H_{ijk}^{t+1}+H_{ij+1k}^{t+1}\right)}{2}\right]\Delta y_{ijk}+w_{ijk}^{t+1/2}\Delta x_{ijk}\Delta y_{ijk}=\frac{\left(\xi_{ij}^{t+1}-\xi_{ij}^{t}\right)\Delta x_{ijk}\Delta y_{ijk}}{\Delta t}$$

$$u_{ijk}^{t+1}H_{ijk}^{t+1}=-w_{ijk}^{t+1/2}\Delta x_{ijk}+\frac{u_{ij+1k}^{t+*}\left(H_{ijk}^{t+1}+H_{ij+1k}^{t+1}\right)}{2}+\frac{\left(\xi_{ij}^{t+1}-\xi_{ij}^{t}\right)\Delta x_{ijk}}{\Delta t} \qquad (60)$$

$$u_{ijk}^{t+1}=\left[-w_{ijk}^{t+1/2}\Delta x_{ijk}+\frac{u_{ij+1k}^{t+*}\left(H_{ijk}^{t+1}+H_{ij+1k}^{t+1}\right)}{2}+\frac{\left(\xi_{ij}^{t+1}-\xi_{ij}^{t}\right)\Delta x_{ijk}}{\Delta t}\right]\Big/H_{ijk}^{t+1}$$

Intermediate layers

$$\left[u_{ijk}^{t+1}H_{ijk}^{t+1}-\frac{u_{ij+1k}^{t+*}\left(H_{ijk}^{t+1}+H_{ij+1k}^{t+1}\right)}{2}\right]\Delta y_{ijk}+w_{ijk}^{t+1/2}\Delta x_{ijk}\Delta y_{ijk}-w_{ijk+1}^{t+1/2}\Delta x_{ijk}\Delta y_{ijk}=0$$

$$u_{ijk}^{t+1}H_{ijk}^{t+1}=\left(w_{ijk+1}^{t+1/2}-w_{ijk}^{t+1/2}\right)\Delta x_{ijk}+\frac{u_{ij+1k}^{t+*}\left(H_{ijk}^{t+1}+H_{ij+1k}^{t+1}\right)}{2} \qquad (61)$$

$$u_{ijk}^{t+1}=\left[\left(w_{ijk+1}^{t+1/2}-w_{ijk}^{t+1/2}\right)\Delta x_{ijk}+\frac{u_{ij+1k}^{t+*}\left(H_{ijk}^{t+1}+H_{ij+1k}^{t+1}\right)}{2}\right]\Big/H_{ijk}^{t+1}$$

Bottom

$$u_{ijk}^{t+1}=\left[w_{ijk+1}^{t+1/2}\Delta x_{ijk}+\frac{u_{ij+1k}^{t+*}\left(H_{ijk}^{t+1}+H_{ij+1k}^{t+1}\right)}{2}\right]\Big/H_{ijk}^{t+1} \qquad (62)$$

Current velocity across a eastern sea boundary (v component = 0):

Surface

$$\left[\frac{u_{ijk}^{t+*}\left(H_{ij-1k}^{t+1}+H_{ijk}^{t+1}\right)}{2}-u_{ij+1k}^{t+1}H_{ijk}^{t+1}\right]\Delta y_{ijk}+w_{ijk}^{t+1/2}\Delta x_{ijk}\Delta y_{ijk}=\frac{\left(\xi_{ij}^{t+1}-\xi_{ij}^{t}\right)\Delta x_{ijk}\Delta y_{ijk}}{\Delta t}$$

$$-u_{ij+1k}^{t+1}H_{ijk}^{t+1}=-w_{ijk}^{t+1/2}\Delta x_{ijk}-\frac{u_{ijk}^{t+*}\left(H_{ij-1k}^{t+1}+H_{ijk}^{t+1}\right)}{2}+\frac{\left(\xi_{ij}^{t+1}-\xi_{ij}^{t}\right)\Delta x_{ijk}}{\Delta t}$$

$$u_{ij+1k}^{t+1}=-\left[-w_{ijk}^{t+1/2}\Delta x_{ijk}-\frac{u_{ijk}^{t+*}\left(H_{ij-1k}^{t+1}+H_{ijk}^{t+1}\right)}{2}+\frac{\left(\xi_{ij}^{t+1}-\xi_{ij}^{t}\right)\Delta x_{ijk}}{\Delta t}\right]\Big/H_{ijk}^{t+1} \qquad (63)$$

$$u_{ij+1k}^{t+1}=\left[w_{ijk}^{t+1/2}\Delta x_{ijk}+\frac{u_{ijk}^{t+*}\left(H_{ij-1k}^{t+1}+H_{ijk}^{t+1}\right)}{2}-\frac{\left(\xi_{ij}^{t+1}-\xi_{ij}^{t}\right)\Delta x_{ijk}}{\Delta t}\right]\Big/H_{ijk}^{t+1}$$

Intermediate layers

$$\left[\frac{u_{ijk}^{t+*}\left(H_{ij-1k}^{t+1}+H_{ijk}^{t+1}\right)}{2}-u_{ij+1k}^{t+1}H_{ijk}^{t+1}\right]\Delta y_{ijk}+w_{ijk}^{t+\frac{1}{2}}\Delta x_{ijk}\Delta y_{ijk}-w_{ijk+1}^{t+\frac{1}{2}}\Delta x_{ijk}\Delta y_{ijk}=0$$

$$-u_{ij+1k}^{t+1}H_{ijk}^{t+1}=w_{ijk+1}^{t+\frac{1}{2}}\Delta x_{ijk}-w_{ijk}^{t+\frac{1}{2}}\Delta x_{ijk}-\frac{u_{ijk}^{t+*}\left(H_{ij-1k}^{t+1}+H_{ijk}^{t+1}\right)}{2} \tag{64}$$

$$u_{ij+1k}^{t+1}=\left[\left(w_{ijk}^{t+\frac{1}{2}}-w_{ijk+1}^{t+\frac{1}{2}}\right)\Delta x_{ijk}+\frac{u_{ijk}^{t+*}\left(H_{ij-1k}^{t+1}+H_{ijk}^{t+1}\right)}{2}\right]\bigg/H_{ijk}^{t+1}$$

Bottom

$$u_{ij+1k}^{t+1}=\left[-w_{ijk+1}^{t+\frac{1}{2}}\Delta x_{ijk}+\frac{u_{ijk}^{t+*}\left(H_{ij-1k}^{t+1}+H_{ijk}^{t+1}\right)}{2}\right]\bigg/H_{ijk}^{t+1} \tag{65}$$

Current velocity across a southern sea boundary (u component = 0):

Surface

$$\left[v_{ijk}^{t+1}H_{ijk}^{t+1}-\frac{v_{i+1jk}^{t+*}\left(H_{ijk}^{t+1}+H_{i+1jk}^{t+1}\right)}{2}\right]\Delta y_{ijk}+w_{ijk}^{t+\frac{1}{2}}\Delta x_{ijk}\Delta y_{ijk}=\frac{\left(\xi_{ij}^{t+1}-\xi_{ij}^{t}\right)\Delta x_{ijk}\Delta y_{ijk}}{\Delta t}$$

$$v_{ijk}^{t+1}H_{ijk}^{t+1}=-w_{ijk}^{t+\frac{1}{2}}\Delta x_{ijk}+\frac{v_{i+1jk}^{t+*}\left(H_{ijk}^{t+1}+H_{i+1jk}^{t+1}\right)}{2}+\frac{\left(\xi_{ij}^{t+1}-\xi_{ij}^{t}\right)\Delta x_{ijk}}{\Delta t} \tag{66}$$

$$v_{ijk}^{t+1}=\left[-w_{ijk}^{t+\frac{1}{2}}\Delta x_{ijk}+\frac{v_{i+1jk}^{t+*}\left(H_{ijk}^{t+1}+H_{i+1jk}^{t+1}\right)}{2}+\frac{\left(\xi_{ij}^{t+1}-\xi_{ij}^{t}\right)\Delta x_{ijk}}{\Delta t}\right]\bigg/H_{ijk}^{t+1}$$

Intermediate layers

$$\left[v_{ijk}^{t+1}H_{ijk}^{t+1}-\frac{v_{i+1jk}^{t+*}\left(H_{ijk}^{t+1}+H_{i+1jk}^{t+1}\right)}{2}\right]\Delta y_{ijk}+w_{ijk}^{t+\frac{1}{2}}\Delta x_{ijk}\Delta y_{ijk}-w_{ijk+1}^{t+\frac{1}{2}}\Delta x_{ijk}\Delta y_{ijk}=0$$

$$v_{ijk}^{t+1}H_{ijk}^{t+1}=\left(w_{ijk+1}^{t+\frac{1}{2}}-w_{ijk}^{t+\frac{1}{2}}\right)\Delta x_{ijk}+\frac{v_{i+1jk}^{t+*}\left(H_{ijk}^{t+1}+H_{i+1jk}^{t+1}\right)}{2} \tag{67}$$

$$v_{ijk}^{t+1}=\left[\left(w_{ijk+1}^{t+\frac{1}{2}}-w_{ijk}^{t+\frac{1}{2}}\right)\Delta x_{ijk}+\frac{v_{i+1jk}^{t+*}\left(H_{ijk}^{t+1}+H_{i+1jk}^{t+1}\right)}{2}\right]\bigg/H_{ijk}^{t+1}$$

Bottom

$$v_{ijk}^{t+1} = \left[w_{ijk+1}^{t+1/2} \Delta x_{ijk} + \frac{v_{i+1jk}^{t+*}\left(H_{ijk}^{t+1} + H_{i+1jk}^{t+1} \right)}{2} \right] \Big/ H_{ijk}^{t+1} \qquad (68)$$

Current velocity across a notthern sea boundary (u component = 0):

Surface

$$\left[\frac{v_{ijk}^{t+*}\left(H_{i-1jk}^{t+1} + H_{ijk}^{t+1} \right)}{2} - v_{i+1jk}^{t+1} H_{ijk}^{t+1} \right] \Delta y_{ijk} + w_{ijk}^{t+1/2} \Delta x_{ijk} \Delta y_{ijk} = \frac{\left(\xi_{ij}^{t+1} - \xi_{ij}^{t} \right) \Delta x_{ijk} \Delta y_{ijk}}{\Delta t}$$

$$-v_{i+1jk}^{t+1} H_{ijk}^{t+1} = -w_{ijk}^{t+1/2} \Delta x_{ijk} - \frac{v_{ijk}^{t+*}\left(H_{i-1jk}^{t+1} + H_{ijk}^{t+1} \right)}{2} + \frac{\left(\xi_{ij}^{t+1} - \xi_{ij}^{t} \right) \Delta x_{ijk}}{\Delta t} \qquad (69)$$

$$v_{i+1jk}^{t+1} = -\left[-w_{ijk}^{t+1/2} \Delta x_{ijk} - \frac{v_{ijk}^{t+*}\left(H_{ij-1k}^{t+1} + H_{ijk}^{t+1} \right)}{2} + \frac{\left(\xi_{ij}^{t+1} - \xi_{ij}^{t} \right) \Delta x_{ijk}}{\Delta t} \right] \Big/ H_{ijk}^{t+1}$$

$$v_{i+1jk}^{t+1} = \left[w_{ijk}^{t+1/2} \Delta x_{ijk} + \frac{v_{ijk}^{t+*}\left(H_{ij-1k}^{t+1} + H_{ijk}^{t+1} \right)}{2} - \frac{\left(\xi_{ij}^{t+1} - \xi_{ij}^{t} \right) \Delta x_{ijk}}{\Delta t} \right] \Big/ H_{ijk}^{t+1}$$

Intermediate layers

$$\left[\frac{v_{ijk}^{t+*}\left(H_{i-1jk}^{t+1} + H_{ijk}^{t+1} \right)}{2} - v_{i+1jk}^{t+1} H_{ijk}^{t+1} \right] \Delta y_{ijk} + w_{ijk}^{t+1/2} \Delta x_{ijk} \Delta y_{ijk} - w_{ij+1k}^{t+1/2} \Delta x_{ijk} \Delta y_{ijk} = 0$$

$$-v_{i+1jk}^{t+1} H_{ijk}^{t+1} = w_{ijk+1}^{t+1/2} \Delta x_{ijk} - w_{ijk}^{t+1/2} \Delta x_{ijk} - \frac{v_{ijk}^{t+*}\left(H_{i-1jk}^{t+1} + H_{ijk}^{t+1} \right)}{2} \qquad (70)$$

$$v_{i+1jk}^{t+1} = \left[\left(w_{ijk}^{t+1/2} - w_{ijk+1}^{t+1/2} \right) \Delta x_{ijk} + \frac{v_{ijk}^{t+*}\left(H_{i-1jk}^{t+1} + H_{ijk}^{t+1} \right)}{2} \right] \Big/ H_{ijk}^{t+1}$$

Bottom

$$v_{i+1jk}^{t+1} = \left[-w_{ijk+1}^{t+1/2} \Delta x_{ijk} + \frac{v_{ijk}^{t+*}\left(H_{i-1jk}^{t+1} + H_{ijk}^{t+1} \right)}{2} \right] \Big/ H_{ijk}^{t+1} \qquad (71)$$

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